Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

The structure of Sidon set systems

(Extended abstract)

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Abstract

A family $\mathcal{F} \subset 2^G$ of subsets of an abelian group G is a *Sidon system* if the sumsets $A + B$ with $A, B \in \mathcal{F}$ are pairwise distinct. Cilleruelo, Serra and the author previously proved that the maximum size $F_k(n)$ of a Sidon system consisting of k-subsets of the first n positive integers satisfies $C_k n^{k-1} \leq F_k(n) \leq \binom{n-1}{k-1} + n - k$ for some constant C_k only depending on k. We close the gap by proving an essentially tight structural result that in particular implies $F_k(n) \geq (1 - o(1))\binom{n}{k-1}$. We also use this to establish a result about the size of the largest Sidon system in the binomial random family $\binom{[n]}{k}_p$. Extensions to h-fold sumsets for any fixed $h \geq 3$ are also obtained.

DOI: `https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-114`

Introduction and main results

A subset A of an abelian group G is a *Sidon set* if the twofold sums of elements in A are pairwise distinct. The study of Sidon sets in the integers is a classical topic in additive number theory, see for instance the survey of O'Bryant [8]. A topic of particular interest is to determine the maximum size of a Sidon set contained in the first n positive integers. Seminal results of Erdős and Turán [5] concerning the upper bound, as well as Ruzsa [9], Bose [2] and Singer [11] for the lower bound established the following result.

Theorem 1 ([5, 11, 2, 9]). *A maximum size Sidon set $A \subset [n]$ satisfies $|A| = (1 \pm o(1))\sqrt{n}$.*

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Interestingly, it is still an open problem to establish the lower order behavior of this cardinality. The main lower order term in the upper bound stood at $n^{1/4}$ since 1969, due to Lindström [7], but the leading constant has recently been pushed below 1 due to Balogh, Füredi and Souktik [1]. The main question, whether the lower order term diverges or not, is still wide open.

One can naturally extend the notion of a Sidon set to set systems. Recall that the *sumset* (or *Minkowski sum*) of two sets A and B is defined as

$$A + B = \{a + b : a \in A, b \in B\}.$$

For an integer $h \geq 2$, we will often write hA as shorthand for the sumset $(h-1)A + A$. In [3], Cilleruelo, Serra and the author defined the following notion of a *Sidon system.*

Definition 2. Let $\mathcal{F} \subset 2^G$ be a family of subsets of an abelian group G. Then $\mathcal{F}$ is a *Sidon system* if for any $A, B, C, D \in \mathcal{F}$ it holds that

$$A + B = C + D \iff \{A, B\} = \{C, D\}.$$

So a Sidon set is just a Sidon system composed entirely of singleton sets. Another way to interpret Sidon systems in an abelian group G is as Sidon sets in the abelian monoid of subsets of G together with the sumset operation.

The size and structure of large Sidon systems of k-sets in $[n]$. In [3] the authors established the following analogue to Theorem 1. We write $F_k(n)$ for the maximum cardinality of a Sidon system composed entirely of k-element subsets of $[n]$.

Theorem 3 ([3])**.** *Let $n > k \geq 2$ be positive integers. Then there exists a constant C_k only depending on k such that*

$$C_k n^{k-1} \leq F_k(n) \leq \binom{n-1}{k-1} + n - k.$$

The major problem left open in [3] was to conclude whether the upper bound in Theorem 3 is asymptotically correct. In fact, a case analysis in the specific case of $k = 3$ did establish this fact. Actually, the authors formulated a stronger conjecture on the structure of Sidon systems, motivated by the proof of the upper bound in Theorem 3.

For integers $n > k \geq 2$, define the set system

$$\binom{[n]}{k}_0 = \{A \subset \{0, 1, \ldots, n\} : |A| = k,\, 0 \in A\}.$$

Then the following conjecture was posed implicitly in [3].

Conjecture 4 ([3])**.** *Let $n > k \geq 3$, and suppose $\mathcal{F} \subset \binom{[n]}{k}$ is any family of k-subsets of the first n integers such that for every $A \in \binom{[n]}{k}_0$ it holds that*

$$|\{x \in \mathbb{Z} : A + x \in \mathcal{F}\}| \leq 1. \tag{1}$$

Then one can remove $o(n^{k-1})$ sets from $\mathcal{F}$ to make it a Sidon system. In particular, by starting with any family that satisfies Eq. (1) *with equality,*

$$F_k(n) \sim \frac{n^{k-1}}{(k-1)!}.$$

As mentioned above, a motivation for Conjecture 4 is the following observation which is one of the main ideas going into proving the upper bound in Theorem 3. For distinct $A, B \in \binom{[n]}{k}_0$ such that $x+A, y+A, u+B, v+B$ are pairwise distinct sets in a Sidon system $\mathcal{F}$, we must have $|x-y| \neq |u-v|$. Since the minimum element of any set in $\binom{[n]}{k}$ can be at most $n-k+1$, their positive differences must lie in $[n-k]$. Hence, after starting with a set system $\mathcal{F}$ as described in Conjecture 4, one can only add at most $n-k$ additional sets to it before it necessarily contains a violation to the Sidon condition. If $k \geq 3$, we see that n is negligible when compared to n^{k-1}, and so here these additional sets can be ignored. The same is not true for $k=2$, and in fact, while the first part of Conjecture 4 holds here, the second does not: In [3] the authors showed that the family

$$\mathcal{F} = \{\{1, n-i\} + \{0, i\} : i = 1, \ldots, n-1\}$$

is a Sidon system. We see that every set in $\binom{[n-1]}{2}_0$ that has a translation in $\mathcal{F}$ except for $\{0, n-1\}$ in fact has two of them. It is also not difficult to check that the size of this family matches the upper bound given by Theorem 3.

As our first result, we resolve Conjecture 4 in the affirmative. Recall that for an integer $h \geq 2$, a subset $A \subset G$ of an abelian group G is called a *B_h-set* if for any $a_1, \ldots, a_h, b_1, \ldots, b_h \in A$ it holds that

$$a_1 + \cdots + a_h = b_1 + \cdots + b_h \iff \{a_1, \ldots, a_h\} = \{b_1, \ldots, b_h\} \text{ as multisets.}$$

This generalizes the notion of Sidon sets by observing that Sidon sets are B_2-sets. We prove the following result.

Theorem 5. *For any positive integer k, there exists an integer $\ell(k) = \ell$ such that the following holds. Let $A, B, C, D \subset \mathbb{R}$ be B_ℓ-sets of cardinality k all having the same minimal element. Then*

$$A + B = C + D \iff \{A, B\} = \{C, D\}.$$

Note that Theorem 5 indeed implies Conjecture 4 by the following argument. Any set in $\binom{[n]}{k}_0$ that is not a B_ℓ-set for some ℓ corresponds to a solution to a system of linear equations

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_k \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = \mathbf{0},$$

with $\lambda_i \in \mathbb{Z}$, $\sum |\lambda_i| \leq 2\ell$ and such that there are at least two indices $0 < i < j \leq k$ with $\lambda_i, \lambda_j \neq 0$. In particular, the matrix on the left-hand side has rank 2, and so there are at most n^{k-2} solutions to this system of linear equations in $[n]$. Since there are clearly at most $(2\ell)^k = O_k(1)$ such matrices, we see that $\binom{[n]}{k}_0$ contains $O_k(n^{k-2})$ non-B_ℓ-sets for any ℓ only depending on k, so one can remove their representatives to obtain a Sidon system.

In fact, the following stronger version of Theorem 5 is proved.

Proposition 6. *For any positive integer k and h, there exists an integer $\ell(k,h) = \ell$ such that the following holds. Let G be an abelian group, and let $A_1, \dots, A_h, B_1, \dots, B_h \subset G$ be B_ℓ-subsets of cardinality k all sharing an element. If there exist indices $i, j \in [h]$ such that $|A_i \cap B_j| \geq 2$, then*

$$A_1 + \cdots + A_h = B_1 + \cdots + B_h \iff \{A_1, \dots, A_h\} = \{B_1, \dots, B_h\} \text{ as multisets.}$$

This implies Theorem 5 for $h = 2$, since one can show that in any linearly ordered group, this minimum intersection requirement is satisfied, even without assuming the sets to be B_ℓ. The key tool in proving Proposition 6 is following simple statement, which holds in arbitrary abelian groups.

Lemma 7. *Let $A, B, C \subset G$ be subsets of an abelian group G such that A is a Sidon set. Then for any set $X \subset A$ satisfying $|X| > |C|$, it holds that*

$$X + B \subset A + C \implies B \subset C.$$

It would be interesting to find out whether an intersection size of size 1 in Proposition 6 is actually possible, and we prove some partial results regarding this.

The largest Sidon system in $\binom{[n]}{k}_p$ and δ-additive families. Recall that the binomial random family $\binom{[n]}{k}_p$ is defined such that every k-set $A \subset [n]$ is contained in $\binom{[n]}{k}_p$ independently with probability p. We write $[n]_p$ for $\binom{[n]}{1}_p$. An interesting question is to study a *sparse random* analogue of determining bounds on $F_k(n)$. That is, instead of investigating the size of the largest Sidon system in $\binom{[n]}{k}$, what happens if we do this in $\binom{[n]}{k}_p$? The Sidon set equivalent of this question was answered by Kohayakawa, Lee, Rödl and Samotij in [6] and they discovered an interesting phase transition. Essentially, as long as $p = o(n^{-1/3})$, the expected number of quadruples violating the Sidon set condition is negligible when compared to the expected size of the random set, and hence standard concentration bounds tell us that the size of the largest Sidon subset will be the same as the size of the random set. For p in the range between $n^{-1/3}$ and constant, the situation is similar to that in $[n]$, that is, the size of the largest Sidon subset is approximately the square root of np, the size of the random set. This range can be seen as an example of the *transference principle* (cf. [4, 10]) that says that results in the dense setting can be moved to the sparse random one in appropriate contexts. Since the problem is clearly monotone in nature, the situation when $n^{-2/3} \leq p \leq n^{-1/3}$ is that the largest Sidon subset must stay constant in the exponent at approximately $n^{1/3}$. Let us summarize.

Theorem 8 ([6]). *Let $0 \leq a \leq 1$ be a fixed constant. Suppose $p = p(n) = (1 + o(1))n^{-a}$. There exists a constant $b = b(a)$ such that almost surely the largest Sidon subset of $[n]_p$ has size $n^{b+o(1)}$. Furthermore,*

$$b(a) = \begin{cases} 1 - a, & \text{if } 2/3 \leq a \leq 1, \\ 1/3, & \text{if } 1/3 \leq a \leq 2/3, \\ (1-a)/2, & \text{if } 0 \leq a \leq 1/3. \end{cases}$$

Our second main result establishes a somewhat less nuanced analogue of Theorem 8. It will be helpful to change the language from the absence to the appearance of additive structures.

Definition 9. Let G be an abelian group and suppose $A, B, C, D \subset G$ are subsets. We say that (A, B, C, D) forms an *additive quadruple* if $A + B = C + D$, and furthermore, it is called *nontrivial* if $\{A, B\} \neq \{C, D\}$.

Hence, a Sidon system is a family that contains no nontrivial additive quadruples. We can now define a relative version of this concept.

Definition 10. Let G be an abelian group and $\delta > 0$. Then a finite family of subsets $\mathcal{F} \subset 2^G$ is called *δ-additive* if every subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| \geq \delta|\mathcal{F}|$ contains a nontrivial additive quadruple.

Using Theorem 5, we are able to determine the threshold probability for when $\binom{[n]}{k}_p$ is δ-additive.

Theorem 11. *Let $k \geq 2$ be a fixed integer and $\delta \in (0, 1)$. Then there exist constants C, c that only depend on k, δ such that*

$$\lim_{n\to\infty} \mathbf{Pr}\left(\binom{[n]}{k}_p \text{ is } \delta\text{-additive}\right) = \begin{cases} 1, & \text{if } p \geq c/n \\ 0, & \text{if } p \leq C/n \end{cases}.$$

Recalling that $F_k(n) \leq O_k(n^{k-1})$ by Theorem 3, this immediately gives us the following analogue of Theorem 8.

Corollary 12. *Let $k \geq 2$ be a fixed integer. Then there exist constants C, c that only depend on k such that asymptotically almost surely, the largest Sidon system $\mathcal{F} \subset \binom{[n]}{k}_p$ has size*

$$|\mathcal{F}| = \begin{cases} \Theta(n^{k-1}), & \text{if } p \geq C/n \\ \Theta(n^k p), & \text{if } p \leq c/n \end{cases}.$$

In other words, we are essentially always in the regime that one can remove a negligible number of k-subsets in order to transform the random family into a Sidon system comparable to the $p = o(n^{-2/3})$ case for Sidon sets.

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