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Semidegree, edge density and antidirected subgraphs

(Extended abstract)

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Abstract

An oriented graph is called *antidirected* if it has no directed path with 2 edges. We prove that asymptotically, any oriented graph D of minimum semidegree greater than $\frac{k}{2}$ contains every balanced antidirected tree of bounded degree and with k edges, and D also contains every antidirected subdivision H of a sufficiently small complete graph K_h, with a mild restriction on the lengths of the antidirected paths in H replacing the edges of K_h, and with H having a total of k edges.
Further, we address a conjecture of Addario-Berry, Havet, Linhares Sales, Reed and Thomassé stating that every digraph on n vertices and with more than $(k-1)n$ edges contains all antidirected trees with k edges. We show that their conjecture is asymptotically true in oriented graphs for balanced antidirected trees of bounded degree and size linear in n.

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1 Introduction

In extremal graph theory, a typical task is to determine conditions on the minimum or the average degree of a graph G (the 'host graph') which guarantee that G contains some

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specific subgraph. We study this type of question for oriented host graphs and restricting ourselves to finding antidirected subgraphs.

We present three theorems. The first of these relates high semidegree to the existence of balanced antidirected trees, where we call an oriented tree *balanced* if its bipartition classes have the same size.

Theorem 1.1. *For all $\eta \in (0,1)$, $c \in \mathbb{N}$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$, every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)\frac{k}{2}$ contains every balanced antidirected tree T with k edges and with $\Delta(T) \leq (\log(n))^c$.*

The second theorem relates high semidegree to the existence of antidirected subdivisions of complete graphs. For $h, k \in \mathbb{N}$, consider any subdivision H of K_h where each edge $e \in E(K_h)$ is substituted by a path with $g(e)$ edges, with $\sum_{e \in E(K_h)} g(e) = k$, and such that the edges of K_h with $g(e) < 3$ induce a forest in K_h. If H has antidirected orientations, then call any antidirected orientation of H a *long k-edge antidirected subdivision of K_h*.

Theorem 1.2. *For all $\eta \in (0,1)$ there are $n_0 \in \mathbb{N}$, $\gamma > 0$ such that for each $n \geq n_0$, each $k \geq \eta n$ and each $h \leq \gamma\sqrt{n}$ the following holds. Every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)\frac{k}{2}$ contains each long k-edge antidirected subdivision of K_h.*

The third theorem related high edge density to the existence of balanced antidirected trees.

Theorem 1.3. *For all $\eta \in (0,1)$, $c \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $k \geq \eta n$, every oriented graph D on n vertices with more than $(1+\eta)(k-1)n$ edges contains each balanced antidirected tree T with k edges and $\Delta(T) \leq (\log(n))^c$.*

Each of our theorems will be motivated and discussed in one of the sections below (the sections follow the same order we chose for stating the theorems here). We provide a discussion of the context of the results and include a sketch of the proof of each of the results. We refer to [14] for more discussion and full proofs.

2 Paths and trees

Dirac (see [5]) observed that if an undirected connected graph G on at least $k+1$ vertices satisfies $\delta(G) \geq \frac{k}{2}$, then G contains a k-edge path (here and later, k is any natural number, independent of the order of the host graph). Trying to translate this result to oriented graphs, a natural possibility would be to replace the minimum degree by the minimum semidegree $\delta^0(D)$, which is defined as the minimum over all the in- and all the out-degrees of the vertices in D, and to ask for certain oriented paths in D.

Jackson [7] showed that every oriented graph D with $\delta^0(D) > \frac{k}{2}$ contains the directed path on k edges. The first author conjectured [13] that in this result, the directed path can be replaced with any oriented path of the same length. This conjecture is best possible for directed paths [7] and also for antidirected paths: observe that in an ℓ-blow-up of the directed triangle (where each vertex is replaced with ℓ independent vertices), any antidirected path covers at most 2ℓ vertices. We show that the conjecture from [13] is asymptotically true:

Corollary 2.1. *For all $\eta \in (0,1)$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$ every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)\frac{k}{2}$ contains every antidirected path with k edges.*

To see that Corollary 2.1 follows from Theorem 1.1, observe that any path either is balanced, or can be extended by one to become balanced. (In the latter case we apply Theorem 1.1 with a sufficiently smaller η.)

Note that the class of trees considered in Theorem 1.1 is very similar to antidirected paths, not only because of the balancedness, but also because of the bounded degree. In graphs, there is a very well-known result for finding bounded degree trees by Komlós, Sárközy and Szemerédi [10]. It states that asymptotically, every graph of minimum degree larger than $\frac{n}{2}$ contains every spanning tree of maximum degree at most $O(\frac{n}{\log n})$. Recently, this result was extended to digraphs by Kathapurkar and Montgomery [8]. Theorem 1.1 can be considered as a version for smaller antidirected trees of Kathapurkar and Montgomery's result, in oriented graphs.

We actually prove a stronger version of Theorem 1.1, namely Theorem 2.2 below, which allows us to choose where the root of the antidirected tree goes. This more general result will be useful in the proof of Theorem 1.3.

Theorem 2.2. *For all $\eta \in (0,1)$, $c \in \mathbb{N}$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$ the following holds for every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)\frac{k}{2}$, and every balanced antidirected tree T with k edges and $\Delta(T) \leq (\log(n))^c$. For each set $V^* \subseteq V(D)$ with $|V^*| \geq \eta n$ and for each $x \in V(T)$, there is an embedding of T in D with x mapped to V^*.*

Sketch of the proof of Theorem 2.2

Given an oriented graph D and an antidirected tree T fulfilling the conditions of the theorem, we apply the digraph regularity lemma to D to find a partition into a bounded number of clusters C_i. The reduced oriented graph R will have a minimum semidegree similar to the one of D (proportionally). Let $x \in V(T)$ and $V^* \subseteq V(D)$ be given, with $|V^*| \geq \eta n$. Note that at least one cluster C_i contains $\eta|C_i|$ vertices from V^*. Let C^* be one such cluster.

Next, we need the concept of a *connected antimatching*: this is a set M of disjoint edges in D such that every pair of edges in M is connected by an *antidirected walk* or simply *antiwalk*, which is a sequence of edges that alternate direction. The *length* of an antidirected walk is its number of edges, where we count repeated edges once for each time they appear.

We show that the minimum semidegree in the reduced oriented graph R suffices to ensure that R contains a large connected antimatching M. Further, the antiwalks connecting the edges of M have bounded length:

Lemma 2.3 (Lemma 4.8 in [14]). *Let $t \in \mathbb{N}^+$, let D be an oriented graph with $\delta^0(D) \geq t$, and let $w \in V(D)$. Then D has a connected antimatching $M = \{a_i b_i\}_{1 \leq i \leq t}$ of size t, with $w = a_1$, and such that, for every $1 \leq i \leq t$, there is an antiwalk of length at most $8t$ containing $a_i b_i$ and $a_1 b_1$.*

Now we turn to our antidirected tree T. We decompose T into a family $\mathcal{T}$ of small subtrees, connected by a constant number of vertices. This type of decomposition has been widely used lately, appearing for the first time in [3]. We prove that it is possible to assign the trees in $\mathcal{T}$ to edges of M in a way that they will fit comfortably into the corresponding clusters, while respecting the orientations. We let P_i denote the set of trees in $\mathcal{T}$ that are assigned to the clusters associated to the edge $a_i b_i \in M$.

We now embed T as follows. In each step, we embed one small tree $S \in \mathcal{T}$. When we choose a new small tree to embed, we make sure that we keep the embedded part connected in the underlying tree. We embed the first d levels of S into the clusters of an antiwalk W_S in R that starts in the cluster containing the image of the parent of the root of S and ends in $a_i b_i$, if $S \in P_i$. The remaining levels of S are embedded into the clusters corresponding to a_i and b_i.

Since T has bounded maximum degree, the union of the first d levels of the trees in $\mathcal{T}$ is very small, and therefore it is not a problem that the first d levels of each $S \in \mathcal{T}$ are embedded in the connecting antidirected walk W_S. After going through all $S \in \mathcal{T}$, we have embedded all of T. For the full proof see [14]. $\square$

3 Subdivisions and cycles

Mader [11] proved that there is a function $g(h)$ such that every (undirected) graph of minimum degree at least $g(h)$ contains a subdivision of the complete graph K_h. Thomassen [15] showed that a direct translation of this result to digraphs is not true. Mader [12] suggested to replace the subdivision of the complete digraph with the *transitive tournament*, i.e. the tournament without directed cycles:

Conjecture 3.1 (Mader [12]). *There is a function $f(h)$ such that every digraph of minimum outdegree at least $f(h)$ contains a subdivision of the transitive tournament of order h.*

This conjecture is open even for $h = 5$. Aboulker, Cohen, Havet, Lochet, Moura and Thomassé [1] observed that in Conjecture 3.1, the minimum outdegree can be replaced with the minimum semidegree, and the resulting conjecture is equivalent to Conjecture 3.1. Our Theorem 1.2 can be seen as a version of Conjecture 3.1 for oriented graphs and antidirected subdivisions of K_h.

For $h = 3$, the objects found in Theorem 1.2 are antidirected cycles. In the existing literature, there are already a number of results on finding oriented cycles with conditions on the minimum semidegree. We will quickly discuss those related to antidirected cycles.

For an oriented cycle C, the *cycle type* of C is defined as the number of forward edges minus the number of backwards edges of C. Note that antidirected cycles have cycle type

0. Kelly, Kühn and Osthus [9] showed that for each $k \geq 3$ and $\eta > 0$ every large enough n-vertex oriented graph of minimum semidegree at least ηn contains all oriented cycles of length at most k and cycle type 0. Further, $\delta^0(D) \geq \frac{3n}{8} + o(n)$ is enough to find a copy of any oriented cycle of length between 3 and n in an oriented graph D [9]. Both results give (quite different) bounds on the semidegree for antidirected cycles. While in the first result, the cycle is small compared to n, in the second result there are antidirected cycles of any even length. Theorem 1.2 provides us with an intermediate semidegree bound for finding an antidirected cycle of medium length:

Corollary 3.2. *For all $\eta \in (0, 1)$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$, every oriented graph D on n vertices with $\delta^0(D) > (1 + \eta)\frac{k}{2}$ contains any antidirected cycle of length at most k.*

Indeed, this corollary follows from Theorem 1.2 since any antidirected cycle with more than four edges can be expressed as a long antisubdivision of K_3, while antidirected C_4 is guaranteed by the results of [9].

Sketch of the proof of Theorem 1.2

Let D be an oriented graph satisfying the conditions of Theorem 1.2. Let a long k-antisubdivision of K_h be given and remove two consecutive inner vertices (along with all adjacent edges) from one of the long antidirected paths of this antisubdivision. Keep removing two vertices from other long antidirected paths until we are left with an antidirected tree T. Denote by $\mathcal{P}$ the set of long antidirected paths of which we removed vertices.

As in the proof of Theorem 2.2, we find a connected antimatching M in the reduced graph R of D. We embed the branch vertices of the antisubdivision into a pair of clusters B, C, such that BC is some fixed edge of M. We start embedding the long antipaths in the clusters corresponding to edges of M, using the antiwalks given by Lemma 2.3 to move between the matching edges.

The only vertices left are the ones removed at the beginning. Since their neighbours are already embedded in $B \cup C$, they can be embedded in $B \cup C$ by regularity. For all details see [14].

4 Edge density

In 1970, Graham [6] confirmed a conjecture he attributes to Erdős: for every antidirected tree T there is a constant c_T such that every sufficiently large directed graph D on n vertices and with at least $c_T n$ edges contains T. A similar statement is false for other oriented trees [2, 4]. In 1982, Burr [4] gave an improvement of Graham's result: Every n-vertex digraph D with more than $4kn$ edges contains each antidirected tree T on k edges, and provides an example where $(k-1)n$ edges are not sufficient. In 2013, Addario-Berry, Havet, Linhares Sales, Reed and Thomassé [2] formulate the following conjecture.

Conjecture 4.1 (Addario-Berry et al. [2]). *Every n-vertex digraph D with more than $(k-1)n$ edges contains each antidirected tree on k edges.*

Theorem 1.3 implies that Conjecture 4.1 is approximately true in oriented graphs for all balanced antidirected trees of bounded maximum degree.

Sketch of the proof of Theorem 1.3

Given the antidirected tree T and the oriented graph D as in the theorem, we start by finding a non-empty oriented subgraph D' of D where each vertex has either out-degree at least $\frac{k}{2}$ or out-degree 0, and either in-degree at least $\frac{k}{2}$ or in-degree 0 (see Lemma 7.1 in [14]). We construct a new oriented graph D'' consisting of four copies of D', two of them with all edges reversed. Because of the way we put those copies together, D'' will have minimum semidegree greater than $\frac{k}{2}$.

Using Theorem 2.2, we embed T into D'', with the root v of T embedded in one of the copies of D' with the original orientations. Taking a little more care, we can ensure that an edge at v is also embedded in this copy. It is then easy to deduce that all of T is embedded into the same copy. Since $D' \subseteq D$, we proved the statement. For the full proof see [14]. $\square$

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