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A new approach for the Brown-Erdős-Sós problem

(Extended abstract)

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Abstract

The celebrated Brown-Erdős-Sós conjecture states that for every fixed e, every 3-uniform hypergraph with $\Omega(n^2)$ edges contains e edges spanned by $e+3$ vertices. Up to this date all the approaches towards resolving this problem relied on highly involved applications of the hypergraph regularity method, and yet they supplied only approximate versions of the conjecture, producing e edges spanned by $e + O(\log e/ \log\log e)$ vertices. We describe a completely different approach, which reduces the problem to a variant of another well-known conjecture in extremal graph theory. A resolution of the latter would resolve the Brown-Erdős-Sós conjecture up to an absolute additive constant.

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1 Introduction

1.1 Background and previous results

Some of the most well studied problems in extremal combinatorics are those asking which objects are guaranteed to appear in "dense" objects. Among notable examples are Roth's Theorem [18] on 3-term arithmetic progressions in dense sets of integers, and the Kővári-Sós-Turán Theorem [16] on bipartite subgraphs of dense graphs. In this paper we consider

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a question raised by Brown, Erdős and Sós in 1973 [3, 2], which is one of the most famous open problems of this type.

Given an integer $e \geq 3$, one would expect a dense 3-uniform hypergraph (3-graph for short) to contain e edges spanned by a small number of vertices. To quantify this, let (v, e)-*configuration* denote a set of e edges spanned by at most v vertices. The Brown–Erdős–Sós Conjecture (BESC) states that for every fixed $e \geq 3$ and all large enough n, every 3-graph with $\Omega(n^2)$ edges contains an $(e+3, e)$-configuration. Despite a lot of effort over the past 50 years, the BESC is only known to hold for $e = 3$, due to a result of Ruzsa and Szemerédi [21].

Since even the $e = 4$ case of the BESC seems hopeless, it is natural to try to prove approximate versions of the conjecture, namely that 3-graphs with $\Omega(n^2)$ edges contain $(e + f(e), e)$-configurations, for some slowly growing function f. The first result of the above type was obtained by Sárközy and Selkow [22] who showed that every 3-graph with $\Omega(n^2)$ edges contains for every fixed e an $(e + 2 + \lfloor \log_2 e \rfloor, e)$-configuration. This was improved by Solymosi and Solymosi [23] for the special case $e = 10$ from 15 to 14 vertices. A general asymptotic improvement of the result of [23] was obtained recently by Conlon, Gishboliner, Levanzov and Shapira [8], who proved the existence of $(e + O(\log e/ \log \log e), e)$-configurations.

Besides its intrinsic interest, the BESC turned out to be one of the most influential problems in extremal combinatorics. For example, the proof of the case $e = 3$ [21] was one of the first applications of Szemerédi's regularity lemma [24], and further introduced the famous graph removal lemma. One of the main motivations for the development of the celebrated hypergraph regularity method [11, 17, 19, 20, 26] was the hope that it will lead to a resolution of BESC. While this did not materialize, the hypergraph regularity method was instrumental in the latest works [8, 23]. However, although the above proofs rely on highly involved applications of the hypergraph regularity method, it appears that the following natural approximate version of the BESC is beyond their reach.

Conjecture 1.1 (Constant deficiency BESC)**.** *There is an absolute constant d so that for every e and every large enough n, every 3-graph with $\Omega(n^2)$ edges contains an $(e + d, e)$-configuration.*

1.2 A new approach for Conjecture 1.1

Our aim in this paper is to reduce Conjecture 1.1 to a problem involving graphs. Let us denote by $\mathrm{ex}(n, H)$ the maximum number of edges in an n vertex graph not containing a copy of H as a subgraph. The Kővári-Sós-Turán Theorem [16] which we mentioned above, states that for every fixed $t \leq s$, we have $\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/t})$ where $K_{s,t}$ is the complete bipartite graph with parts of size t and s. This bound is known to be tight for large s, see [4] for recent progress and references. One of the main research directions in extremal graph theory is to obtain better bounds for sparser bipartite graphs. One such problem was raised by Erdős [9], who conjectured that if H is a t-degenerate bipartite graph then $\mathrm{ex}(n, H) = O(n^{2-1/t})$. While there are some approximate results towards this conjecture [1, 10, 13, 15], the question is open even for $t = 2$. Note that in general, the conjectured bound $O(n^{2-1/t})$ for t-degenerate bipartite graphs cannot be improved since the aforementioned $K_{s,t}$ is t-degenerate. In particular, the bound is tight for every t-degenerate H which contains a copy of $K_{s,t}$. In light of this, Conlon [5] conjectured that if we assume that a t-degenerate bipartite graph H has no $K_{t,t}$ then we have $\text{ex}(n, H) = O(n^{2-1/t-\delta})$ for some $\delta = \delta(H) > 0$. Lending plausibility to this conjecture, Sudakov and Tomon [25] showed that if all vertices in one of the parts of H have degree at most t but H has no $K_{t,t}$ then $\text{ex}(n, H) = o(n^{2-1/t})$. For $t = 2$ Conlon's conjecture can be stated as:

Conjecture 1.2 (Conlon [5])**.** *For every 2-degenerate C_4-free bipartite graph H there exists a constant $\delta = \delta(H) > 0$ such that*

$$ex(n, H) = O(n^{3/2-\delta}) .$$

There are several results supporting Conjecture 1.2. For example, Conlon and Lee [7] proved that if H is a bipartite graph so that each vertex in one of H's sides has maximum degree 2 (such a graph is clearly 2-degenerate) and H is C_4-free then $\text{ex}(n, H) = O(n^{3/2-\delta})$ for some $\delta = \delta(H) > 0$. Further results in this direction were obtained in [6, 14].

Let $\mathcal{H}_{k,t}$ be the family of 2-degenerate graphs on k vertices and $2k - t$ edges. We raise the following weaker version of Conjecture 1.2.

Conjecture 1.3. *There are absolute constants t, k_0 such that for every $k \geq k_0$ and large enough n, every graph with $\Omega(n^{3/2})$ edges contains a copy of* **some** *$H \in \mathcal{H}_{k,t}$.*

Let us briefly explain why Conjecture 1.3 is indeed weaker than Conjecture 1.2. It is not hard to see that for every t and large enough k, the family $\mathcal{H}_{k,t}$ contains C_4-free graphs (see Claim 3.1). Conjecture 1.2 then states that if G has $\Omega(n^{3/2})$ edges then G should contain a copy of **every** $H \in \mathcal{H}_{k,t}$ which is C_4-free, while Conjecture 1.3 only asks G to contain a copy of some $H \in \mathcal{H}_{k,t}$. Note also that Conjecture 1.3 is weaker than the statement that for every $k \geq k_0$ we have $\text{ex}(n, H) = o(n^{3/2})$ for some $H \in \mathcal{H}_{k,t}$, which is itself weaker than Conjecture 1.2.

Our main result in this paper is the following alternative approach for resolving Conjecture 1.1.

Theorem 1.4. *Conjecture 1.3 implies Conjecture 1.1.*

Before turning to the proof of Theorem 1.4, we mention that it might very well be the case that in Conjecture 1.3 we can replace the lower bound $\Omega(n^{3/2})$ by $\Omega(n^{3/2-\delta})$ for some $\delta = \delta(k) > 0$. Indeed, this bound is implied by Conjecture 1.2. It is not hard to see that in this case the proof of Theorem 1.4 would give that for some absolute constant d and for every e there is $\varepsilon = \varepsilon(e) > 0$ so that one can find $(e + d, e)$-configurations in every 3-graph with $n^{2-\varepsilon}$ edges. Such a result would be an approximate version of a conjecture suggested by Gowers and Long [12], stating that 3-graphs with $n^{2-\varepsilon}$ edges contain $(e+4, e)$-configurations.

2 Proof of Theorem 1.4

To avoid confusion, we will refer to edges of a 3-graph as hyperedges. Fix $e \geq 3$ and let $\mathcal{G}$ be a 3-graph with n vertices and $\Omega(n^2)$ hyperedges. We will rely on the well known observation that in the context of the BESC one can assume that $\mathcal{G}$ is linear and 3-partite on vertex sets $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. We now apply a variant of the construction of Solymosi and Solymosi [23]. Given $\mathcal{G}$, define an auxiliary bipartite multigraph G' as follows. Set $V(G') = (A, B)$ where $A = \binom{\mathcal{A}}{2}$ and $B = \binom{\mathcal{B}}{2}$. For two vertices $\{a_1, a_2\} \in A$ and $\{b_1, b_2\} \in B$ put an edge between them if there is a $c \in \mathcal{C}$ so that a_1b_1c and a_2b_2c are hyperedges of $\mathcal{G}$, and (independently) put an edge between them if there is a $c' \in \mathcal{C}$ such that a_1b_2c' and a_2b_1c' are hyperedges of $\mathcal{G}$. Since $\mathcal{G}$ is linear, each pair of vertices in G' are connected by at most 2 edges. If we let $d(c)$ denote the degree of a vertex $c \in \mathcal{C}$ in $\mathcal{G}$ then

$$|E(G')| = \sum_{c \in \mathcal{C}} \binom{d(c)}{2} \geq |\mathcal{C}| \binom{\frac{1}{|\mathcal{C}|}\sum_{c\in\mathcal{C}} d(c)}{2} = |\mathcal{C}| \binom{|E(\mathcal{G})|/|\mathcal{C}|}{2} \geq \frac{|E(\mathcal{G})|^2}{4|\mathcal{C}|}.$$

Since $e(\mathcal{G}) = \Omega(n^2)$, $|\mathcal{C}| \leq n$, and $|V(G')| \leq n^2$, we obtain $|E(G')| = \Omega(|V(G')|^{3/2})$. Since, as noted above, each pair of vertices in G' are connected by at most 2 edges, G' has a simple subgraph G which also contains $\Omega(|V(G)|^{3/2})$ edges. Therefore, if k_0 and t are the constants from Conjecture 1.3 and n is large enough, then we may assume the following.

Observation 2.1. *For every $k_0 \leq k \leq e$, the graph G contains a 2-degenerate bipartite graph F on k vertices with at least $2k - t$ edges.*

We would now like to understand what kind of (v, e)-configuration in $\mathcal{G}$ we get by "unpacking" each of the graphs F in Observation 2.1. Optimistically, if $v_1, \ldots, v_k$ is the ordering of $V(F)$ certifying its 2-degeneracy, then every time we add a vertex v_i to $v_1, \ldots, v_{i-1}$ of degree 2 to the previous vertices, we expect to get 4 new vertices in $\mathcal{G}$; these are c_1, c_2 and either a_1, a_2 (if $v_i \in A$) or b_1, b_2 (if $v_i \in B$). We also expect to get 4 new hyperedges in $\mathcal{G}$; these are the 4 hyperedges that correspond to the 2 new edges in G that connect v_i to 2 of the vertices $v_1, \ldots, v_{i-1}$. If this holds for all but a bounded number of F's vertices, then we will get a $(4k, 4k - O_k(1))$ configuration, hence taking $k \approx e/4$ would finish the proof. Unfortunately, we do not know how to prove such a statement, since in certain cases (see below) some of the 4 vertices/hyperedges might have already appeared when adding one of the previous vertices v_j. Instead, the main idea in Lemma 2.2 below is to show that F gives rise to a $(e' + d, e')$-configuration, so that if e' is not very close to $4k$ (as in the optimistic analysis above) then we have $d \leq 0$. It is then easy to show how repeated applications of Lemma 2.2 give Theorem 1.4. In what follows G and $\mathcal{G}$ are those we discussed above.

Lemma 2.2. *Let $k \geq t \geq 4$ be integers, and suppose F is a 2-degenerate subgraph of G with k vertices and $2k - t$ edges. Then $\mathcal{G}$ contains a subgraph $\mathcal{F}$ such that*

(1) $|V(\mathcal{F})| - 4t \leq |E(\mathcal{F})| \leq 4k$, and

(2) Either $|E(\mathcal{F})| \geq 4k - 10^4t^3$ or $|E(\mathcal{F})| \geq |V(\mathcal{F})| > 0$.

For the proof of Lemma 1.4 we refer the reader to the full version of the paper. We will now show how to derive Theorem 1.4 from Lemma 2.2. Assuming Conjecture 1.3 holds with constants t, k_0 we show that Conjecture 1.1 holds with $d = \max\{24k_0, 3(4t + 10^4t^3)\}$. Indeed, we claim that for every $0 \le e' \le e$ we can find e' hyperedges in $\mathcal{G}$ spanned by at most $e'+d$ vertices. If $e' \le \max\{8k_0, 4t+10^4t^3\}$, we just take e' arbitrary hyperedges from $\mathcal{G}$. For larger e' we apply Lemma 2.2 with the above t and with $k = \lfloor e'/4 \rfloor \ge k_0$ (by Observation 2.1 we know that G contains an F with these parameters). If the lemma returns a configuration $\mathcal{F}'$ whose number of edges satisfies $e' - 10^4t^3 - 4 \le |E(\mathcal{F}')| \le e'$ (and is on at most $e' + 4t$ vertices), we just add to $\mathcal{F}'$ arbitrarily chosen $e' - |E(\mathcal{F}')| \le 10^4t^3 + 4$ hyperedges to get a set of e' edges on at most $e' + d$ vertices. Otherwise, we have $|E(\mathcal{F}')| \ge |V(\mathcal{F}')| > 0$ so we can remove $\mathcal{F}'$ from $\mathcal{G}$ and then restart the process with $e'' = e' - |E(\mathcal{F}')|$ (the 3-graph $\mathcal{G} \setminus \mathcal{F}$ still has $\Omega(n^2)$ hyperedges assuming n is large). We will obtain a set $\mathcal{F}''$ of e'' hyperedges on at most $e'' + d$ vertices, and can then return $\mathcal{F}'' \cup \mathcal{F}'$ as the set of e' hyperedges on at most $e' + d$ vertices.

3 C_4-free graphs in $\mathcal{H}_{k,t}$

We say that a graph is *exactly-*$(2,t)$*-degenerate* if it can be obtained from a set of t isolated vertices by repeatedly adding new vertices of degree exactly 2. Note that every exactly-$(2,t)$-degenerate graph belongs to $\mathcal{H}_{k,t}$. The following claim shows that $\mathcal{H}_{k,t}$ contains not only C_4-free graphs, but in fact graphs of arbitrary large girth.

Claim 3.1. *For every g there is $t = t(g)$ so that for every $k \ge t$, there is a k-vertex exactly-$(2,t)$-degenerate bipartite graph of girth at least g.*

Proof. We claim that starting with an independent set of size $t = t(g)$ (to be chosen later), we can repeatedly add vertices so that each k-vertex graph in the sequence is exactly-$(2,t)$-degenerate, bipartite, of girth at least g, and in addition satisfies the following two conditions: (i) it has maximum degree at most 8 and (ii) it has a bipartition into two set of sizes $\lceil k/2 \rceil$ and $\lfloor k/2 \rfloor$. The initial independent set under a balanced bipartition clearly satisfies these two conditions, so let us show how to add a vertex and maintain them. Suppose the graph has $k-1$ vertices and bipartition into sets A, B satisfying $|A| \le |B|$. Since it has maximum degree at most 8, it contains $O(k)$ pairs of vertices connected by a path of length at most $g-2$. Since the average degree of the vertices in B is less than 4, at least half the vertices have degree at most 7. Hence, at least $\binom{(k-1)/4}{2} \ge \frac{k^2}{50}$ of the pairs of vertices in B both have degree at most 7. Assuming t is large enough so that $k \ge t$ satisfies $\frac{k^2}{50} - O(k) > 1$, we thus have a pair of vertices $u, v \in B$ so that both of them have degree at most 7 and there is no path of length at most $g-2$ connecting them. Hence, we can add a new vertex to A and connect it to u and v. $\square$

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