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On universal singular exponents in equations with one catalytic parameter of order one

(Extended abstract)

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Abstract

Equations with one catalytic variable and one univariate unkown, also known as *discrete difference equations of order one*, form a familly of combinatorially relevant functional equations first discussed in full generality by Bousquet-Mélou and Jehanne (2006) who proved that their power serie solutions are algebraic. Drmota, Noy and Yu (2022) recently showed that in the non linear case the singular expansions of these series have a universal dominant term of order $3/2$, as opposed to the dominant square root term of generic $\mathbb{N}$-algebraic series. Their direct analysis of the cancellation underlying this behavior is a tour de force of singular analysis. We show that the result can instead be given a straightforward explanation by showing that the derivative of the solution series conforms to the standard square root singular behavior. Consequences also include an atypical, but generic in this situation, $n^{5/4}$ asymptotic behavior for the cumulated values of the underlying catalytic parameter.

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Context and known results. Our interest is in families of bigraded combinatorial structures $\mathscr{F} = (\mathscr{F}_{n,k})_{n,k\geqslant 0}$ whose bivariate generating series satisfiy a so-called *equation with one catalytic variable and one univariate unknown* [3], or *discrete difference equation of order one* [1]. Many examples of such combinatorial structures have surfaced over the

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last fifty years in the combinatorial literature, among which proeminent topics of recent interest like *lattice paths*, *rooted planar maps*, *two-stack-sortable permutations*, *normal linear lambda terms*, *intervals of the Tamari lattice*, *figting fish* or *tree parking functions* (see *e.g.* references in [1, 3, 7, 8, 5]).

Catalytic equations as a generic class of functional equations were notably studied by Bousquet-Mélou and Jehanne [3] who proved that the power series solutions of catalytic equations with one catalytic variable (and arbitrary order) are in general algebraic functions. Further explicit universal results were more recently obtained for order one catalytic equations: complexity issues were discussed by Bostan *et al* [1], while Drmota, Noy and Yu [7] exibited a universal critical exponant governing the polynomial correction in the asymptotic of the coefficients of the solution series (see also Chen [5] for more detailed universality results concerning a slightly more restricted class of functions, and Chapuy [4] for earlier partial results along the same lines as [7]).

Here we concentrate on the generic order one catalytic equation with one variable, as studied by Drmota, Noy and Yu [7]:

$$F(u) = Q\left(F(u), \tfrac{1}{u}(F(u) - f), u, t\right) \tag{1}$$

where $F(u) \equiv F(u,t) = \sum_{n,k\geqslant 0} |\mathcal{F}_{n,k}| t^n u^k$ and $f \equiv f(t) = F(0,t)$ are respectively the ordinary generating functions of elements of $\mathcal{F}$ and of $\mathcal{f} = \cup_{n\geqslant 0}\mathcal{F}_{n,0}$, with t marking the size n and u marking the secondary parameter k (refered to as the *catalytic parameter*), and where $Q(v,w,u,t) = \sum_{i,j,k,\ell\geqslant 0} q_{i,j,k,\ell} v^i w^j u^k t^\ell$, is assumed[1] to take the form $Q(v,w,u,t) = Q_0(u) + tQ_+(v,w,u,t)$ with Q_0 and Q_+ polynomials with non negative coefficients such that $Q''_{\mathrm{vv}} + Q''_{\mathrm{vw}} + Q''_{\mathrm{ww}} \neq 0$ (non linearity condition), and $Q'_w + Q''_{\mathrm{vu}} \neq 0$ (catalytic condition).

Following Bousquet-Mélou and Jehanne [3], we consider the derivative of Equation (1) with respect to u, as given formally by the standard chain rule for derivation:

$$\begin{aligned}\frac{\partial F}{\partial u}(u) \quad = \quad & \left(\frac{\partial Q}{\partial v}\left(F(u), \tfrac{1}{u}(F(u)-f), u, t\right) + \frac{\partial Q}{\partial w}\left(F(u), \tfrac{1}{u}(F(u)-f), u, t\right)\tfrac{1}{u}\right)\frac{\partial F}{\partial u}(u) \\ & + \left(\frac{\partial Q}{\partial u}\left(F(u), \tfrac{1}{u}(F(u)-f), u, t\right) - \frac{\partial Q}{\partial w}\left(F(u), \tfrac{1}{u}(F(u)-f), u, t\right)\tfrac{1}{u^2}(F(u)-f)\right).\end{aligned} \tag{2}$$

Under our assumptions on Q, upon extracting coefficients of successives powers of t, the equation

$$U = U\frac{\partial Q}{\partial v}\left(F(U), \tfrac{1}{U}(F(U)-f), U, t\right) + \frac{\partial Q}{\partial w}\left(F(U), \tfrac{1}{U}(F(U)-f), U, t\right)$$

extracted from the first line of Equation (2) is seen to have a unique power series solution $U \equiv U(t)$ in $t \cdot \mathbb{Q}[t]$, and $\frac{\partial F}{\partial u}(U(t))$ is a well defined power series in t. The substitution

[1]Actually our analysis applies in a slighltly more general analytic setting, provided Q has non negative coefficients and the equation is non degenerate, non linear and catalytic, but we stick for simplicity with the polynomiality assumptions used in [3, 7] which covers most known examples.

$u = U$ then cancels the first line of Equation (2), and therefore also its second line. This implies that the series U, $V \equiv V(t) = F(U)$, and $W \equiv W(t) = \frac{F(U)-f}{U}$ satisfy the systems

$$\left\{\begin{array}{rcl} V &=& Q(V,W,U,t), \\ U &=& U\frac{\partial Q}{\partial v}(V,W,U,t) + \frac{\partial Q}{\partial w}(V,W,U,t), \\ 0 &=& \frac{\partial Q}{\partial u}(V,W,U,t) - \frac{\partial Q}{\partial w}(V,W,U,t)\frac{W}{U}, \\ f &=& V-UW, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcl} V &=& Q(V,W,U), \\ U &=& U\frac{\partial Q}{\partial v}(V,W,U) + \frac{\partial Q}{\partial w}(V,W,U), \\ W &=& W\frac{\partial Q}{\partial v}(V,W,U) + \frac{\partial Q}{\partial u}(V,W,U), \\ f &=& V-UW. \end{array}\right. \tag{3}$$

where the second system is obtained, following Drmota, Noy and Yu [7], by replacing Line 3 of the first system by the linear combination $\frac{W}{U}(\text{Line}\,2) + (\text{Line}\,3)$.

This system was used by Drmota, Noy and Yu [7] to derive the asymptotic behavior of the coefficients of the series f under the aforementioned assumption that $Q(v,w,u,t) = Q_0(u) + tQ_+(v,w,u,t)$ where Q_0 and Q_+ are polynomials of $\mathbb{Q}[v,w,u,t]$ with non negative coefficients. Apart in the linear case and in a few other simple degenerate situations discussed in [7], these assumptions imply that the three series V, U, and W are the unique power series solutions of a system of three polynomial equations with non negative coefficients whose dependancy graph is strongly connected: the celebrated Drmota-Lalley-Woods theorem [10, Thm VII.6, p. 489] then immediately yields that these series have a common dominant singular behavior of the square root type:

$$\begin{array}{rcl} V(t) &=& a_V - b_V\sqrt{1-t/\rho} + c_V(1-t/\rho) + d_V(1-t/\rho)^{3/2} + O((1-t/\rho)^2), \\ W(t) &=& a_W - b_W\sqrt{1-t/\rho} + c_W(1-t/\rho) + d_W(1-t/\rho)^{3/2} + O((1-t/\rho)^2), \\ U(t) &=& a_U - b_U\sqrt{1-t/\rho} + c_U(1-t/\rho) + d_U(1-t/\rho)^{3/2} + O((1-t/\rho)^2), \end{array} \tag{4}$$

near their common radius of convergence $\rho > 0$, for positive constants a_V, b_V, a_W, b_W, a_U, and b_U and constants c_V, d_V, c_W, d_W, c_U, and d_U that can be explicitely expressed in terms of ρ, Q and its derivatives.

A first computation with these explicit expressions shows that a systematic cancellation of the square root terms occurs when these singular expansions are pluged in the expression $f(t) = V(t) - W(t)U(t)$, so that f is generically expected to admit a singular expansion with the next possible higher order $3/2$:

$$f = \alpha + \beta(1-t/\rho) - \gamma(1-t/\rho)^{3/2} + O((1-t/\rho)^2).$$

Using higher order expansions given by Drmota-Lalley-Wood theorem, the constant γ can be in turn expressed in terms of higher derivatives of Q. However showing that γ is positive is non trivial (as opposed to the easy statement that it is non negative), and Dmota, Noy and Yu [7] develop a quite delicate analysis to obtain this result, showing that the exponent $3/2$ is indeed universal.

Under standard technical aperiodicity conditions (see the detailed discussion in [7]), classical transfer theorems [10, Thm VI.3, p390] then imply Drmota, Noy and Yu's main result [7, Thm 2] that the coefficients of f behave as $[t^n]f(t) \sim \text{cte}\cdot\rho^{-n}\cdot n^{-5/2}$. This is a beautiful achievement to be compared for instance to the standard universal $\text{cte}\cdot\rho^{-n}\cdot n^{-3/2}$ asymptotic behavior of the coefficients of generating series of irreducible context free structures amenable to the Drmota-Lalley-Wood theorem [10, Thm VII.5, p483].

Our short derivation. The purpose of this note is to make an observation that allows to circumvent the delicate analysis of the cancellation at the heart of Drmota, Noy and Yu's approach, and to give a direct explanation of the universal asymptotic behavior of the coefficients of f. Consider the derivation of Equation (1) with respect to t :

$$\begin{aligned}\tfrac{\partial F}{\partial t}(u) \;&=\; \left(\tfrac{\partial Q}{\partial v}\left(F(u), \tfrac{1}{u}(F(u)-f), u, t\right) + \tfrac{\partial Q}{\partial w}\left(F(u), \tfrac{1}{u}(F(u)-f), u, t\right)\tfrac{1}{u}\right)\tfrac{\partial F}{\partial t}(u) \\ &+\; \left(\tfrac{\partial Q}{\partial w}\left(F(u), \tfrac{1}{u}(F(u)-f), u, t\right)\right)\tfrac{(-1)}{u}\tfrac{\partial f}{\partial t} + \tfrac{\partial Q}{\partial t}\left(F(u), \tfrac{1}{u}(F(u)-f), u, t\right).\end{aligned} \tag{5}$$

In view of the chain rule for derivation, the first line of Equation (5) has the exact same form as the first line of Equation (2) so that it also cancels upon substituting $u = U$. As a consequence, the second line of Equation (5) yields the identity

$$\left(\frac{\partial Q}{\partial w}(V, W, U, t)\right)\tfrac{(-1)}{U}\frac{\partial f}{\partial t} + \frac{\partial Q}{\partial t}(V, W, U, t) = 0. \tag{6}$$

Using again a linear combination, $\frac{1}{U}\frac{\partial f}{\partial t}$ (Line 2 of System (3)) + (Equation (6)), we obtain:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}\frac{\partial Q}{\partial v}(V, W, U, t) + \frac{\partial Q}{\partial t}(V, W, U, t).$$

Upon letting $S \equiv S(t)$ denote the unique formal power series in $\mathbb{Q}[t]$ solution of the equation $S = 1 + S\frac{\partial Q}{\partial v}(V, W, U, t)$, we obtain the larger but completely non negative system

$$\left\{\begin{array}{lcl} V &=& Q(V, W, U, t), \\ S &=& 1 + S\frac{\partial Q}{\partial v}(V, W, U, t), \\ U &=& S\frac{\partial Q}{\partial w}(V, W, U, t), \\ W &=& S\frac{\partial Q}{\partial u}(V, W, U, t), \\ \frac{\partial f}{\partial t} &=& S\frac{\partial Q}{\partial t}(V, W, U, t). \end{array}\right. \tag{7}$$

Observe in particular that apart in a few degenerate cases (which are the same already listed in [7]) the dependancy graph of the four first unknowns $\{F, S, U, W\}$ in the four first equations of System (7) is strongly connected. Hence with the same assumptions as above, the hypotheses of the classical Drmota-Lalley-Woods theorem are satisfied again. As a consequence these four series have a singular expansion of the form $a_x - b_x\sqrt{1 - t/\rho} + O(1 - t/\rho)$ near their common dominant singularity $\rho > 0$, with computable positive constants a_x and b_x specific to each series $x \in \{V, S, W, U\}$.

Our main observation is then that, since singular expansions of the square root type are preserved via finite products and sums, the last equation of our system immediately provides a singular expansion of the square root type for the derivative of f:

$$\frac{\partial f}{\partial t}(t) = S(t)\frac{\partial Q}{\partial t}(V(t), W(t), U(t), t) = \alpha' - \beta'\sqrt{1 - t/\rho} + O(1 - t/\rho).$$

In particular the positivity of β' immediately follows from the positivity of the coefficients of Q and of the various constants a_* and b_*.

Under the usual aperiodicity conditions, this immediately yields that

$$[t^n]\frac{\partial f}{\partial t}(t) \sim \text{cte}\cdot\rho^{-n}\cdot n^{-3/2} \quad \text{and} \quad [t^n]f(t) = \frac{1}{n}[t^{n-1}]\frac{\partial f}{\partial t}(t) \sim \text{cte}\cdot\rho^{-n}\cdot n^{-5/2}.$$

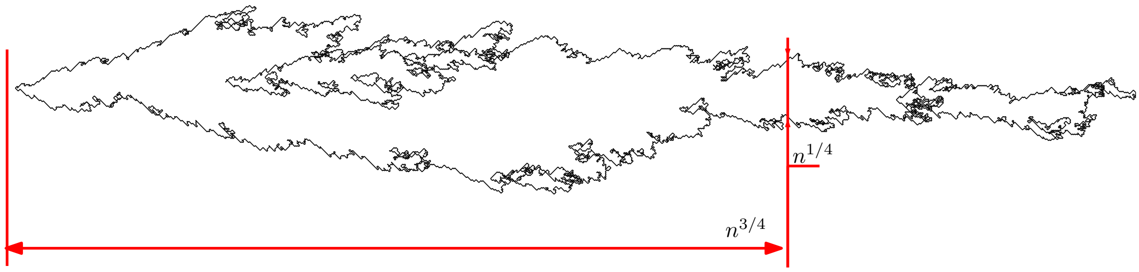

Figure 1: A fighting fish and its average parameters

Cumulated catalytic parameter. Equation like (1) typically arise from a combinatorial recursive decomposition using a so-called catalytic parameter, an auxiliary parameter which characterizes the substructures involved in the decomposition. Refinements of Equations (7) then allow to derive the asymptotic behavior of a number of interesting combinatorial parameters that are closely related to the catalytic parameter: typically the average depth of a node in the decomposition trees with root parameter 0 is of order $n^{3/4}$, and the average value of a random substructure is of order $n^{1/4}$. In this context the basic quantity that governs these behavior is the cumulated value of the catalytic parameter over all decreasing substructures of the structures of size n. This quantity is captured by the series $Z = U \cdot \frac{\partial F}{\partial u}(U)$, whose singular behavior can be derived upon derivating Equation (2) a second time with respect to u and using Equations (7): the series Z satisfies a quadratic equation with coefficients that depends on derivatives of Q evaluated at V, W and U, whose discriminant cancels at first order at the singularity, leading to a dominant term $(1 - t/\rho)^{1/4}$ in the singular expansion.

A concrete example is that of fighting fish, where the cumulated label corresponds to a variant of the area (namely the area in the narrowing columns of the fish), in terms of which the width, as well as the depth of a random point of the boundary can be computed, leading to the results illustrated by Figure 1. The random instance displayed in Figure 1 has size 10000 and was generated using a random sampling algorithm for non separable maps [11] combined with the recent bijection of Duchi and Henriet [8].

Concluding remarks. From a combinatorial point of view, and in accordance with Schützenberger methodology [2], the fact that the derivative of the solution $f(t)$ of an order one equations with one catalytic variable is the solution of a system of polynomial equations with non negative coefficients suggests that if a combinatorial family admits a *first order recursive specification with one catalytic variable*, its derivative family should enjoy a *context free specification* in the sense of [10, Chapter VII.6]. This is the topic of a forthcoming article of Duchi and the author [9].

From an analytic point of view a natural question is to understand if the results can be extended to higher order catalytic equations with one catalytic variable, that are expected to share the same universal asymptotic behavior. A first remarkable achievment in this direction was recently obtained by Drmota and Hainzl for second order equations [6].

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