

THE TURÁN NUMBER OF SURFACES

(EXTENDED ABSTRACT)

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Abstract

We show that there is a constant c such that any 3-uniform hypergraph $\mathcal{H}$ with n vertices and at least $cn^{5/2}$ edges contains a triangulation of the real projective plane as a sub-hypergraph. This resolves a conjecture of Kupavskii, Polyanskii, Tomon, and Zakharov. Furthermore, our work, combined with prior results, asymptotically determines the Turán number of all surfaces.

DOI: <https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-110>

1 Introduction

Turán-type questions are fundamental in the study of extremal combinatorics. Given a fixed r -uniform hypergraph $\mathcal{F}$, its *Turán number* $\text{ex}(n, \mathcal{F})$ is the maximum number of edges in an r -uniform hypergraph $\mathcal{H}$ on n vertices which does not contain $\mathcal{F}$ as a sub-hypergraph. Estimating Turán numbers for hypergraphs remains a largely open problem; we refer the reader to the surveys [1, 3, 9] for a general overview.

In this paper, we investigate a topological variant of this problem. Any r -uniform hypergraph $\mathcal{H}$ may be viewed as an $(r - 1)$ -dimensional simplicial complex whose facets are the edges of $\mathcal{H}$. Similarly, one may ask if any sub-hypergraph of $\mathcal{H}$ is homeomorphic to a given $(r - 1)$ -dimensional simplicial complex X . This topological perspective yields many natural generalizations of graph properties to higher dimensions. For example, one analogue of Hamiltonian cycles in 3-uniform hypergraphs that has received some attention (see [2, 8]) is a spanning sub-hypergraph homeomorphic to the 2-sphere. Additionally,

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one is naturally interested in the following extremal quantity. Let X be a closed $(r - 1)$ -dimensional manifold. Denote by $\text{ex}_{\text{hom}}(n, X)$ the maximum number of edges in a r -uniform hypergraph $\mathcal{H}$ on n vertices such that no sub-hypergraph of $\mathcal{H}$ is homeomorphic (as a simplicial complex) to X . This is the *Turán number* of the topological space X .

As part of his program in high-dimensional combinatorics, Linial [7] asked for the asymptotics of $\text{ex}_{\text{hom}}(n, X)$ when $r \geq 3$. Linial's question was partially motivated by the work of Sós, Erdős, and Brown [11] some decades prior, which showed that $\text{ex}_{\text{hom}}(n, X) = \Theta(n^{5/2})$ when X is the 2-sphere $\mathbb{S}^2$. Linial [7] sketched a new proof of the lower bound $\text{ex}_{\text{hom}}(n, \mathbb{S}^2) = \Omega(n^{5/2})$ which generalized to all closed, connected 2-manifolds X ; this proof is given rigorously in [5, §2]. We call such a 2-manifold a *surface*.

All surfaces fall into one of three categories: the sphere $\mathbb{S}^2$, the connected sum of $g \geq 1$ tori, or the connected sum of $k \geq 1$ real projective planes. Until recently, it was unknown if the lower bound of $n^{5/2}$ was asymptotically tight for the latter two classes. Indeed, Linial [6, 7] repeatedly conjectured a matching upper bound for the torus $\mathbb{T}^2$, i.e. that $\text{ex}_{\text{hom}}(n, \mathbb{T}^2) = O(n^{5/2})$. Kupavskii, Polyanskii, Tomon, and Zakharov [5] proved Linial's conjecture in 2020. Additionally, they showed that if two surfaces X_1, X_2 satisfy $\text{ex}_{\text{hom}}(n, X_i) = O(n^{5/2})$, their connected sum $X_1 \# X_2$ also satisfies $\text{ex}_{\text{hom}}(n, X_1 \# X_2) = O(n^{5/2})$, thereby extending the upper bound $\text{ex}_{\text{hom}}(n, X) = O(n^{5/2})$ to orientable surfaces of the form $X = \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$. They were unable to derive the corresponding result for any non-orientable surfaces, but conjectured that the same bound applies to all surfaces.

Our main result is the resolution of this conjecture. We show that Linial's lower bound is asymptotically tight for the real projective plane $\mathbb{RP}^2$.

Theorem 1.1. *We have $\text{ex}_{\text{hom}}(n, \mathbb{RP}^2) = O(n^{5/2})$.*

By Kupavskii, Polyanskii, Tomon, and Zakharov's result on connected sums, this bound generalizes to all non-orientable surfaces $X = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$. Combining our work with the results of Sós, Erdős, and Brown [11] for the sphere and Kupavskii, Polyanskii, Tomon, and Zakharov [5] for all other orientable surfaces, we completely determine the asymptotics of $\text{ex}_{\text{hom}}(n, X)$ for any surface X .

Theorem 1.2. *Let X be any surface. Then $\text{ex}_{\text{hom}}(n, X) = \Theta(n^{5/2})$, where the constant coefficients may depend on the surface X .*

In the remaining two sections, we sketch the proof of Theorem 1.1. We first describe how to build a hypergraph homeomorphic to $\mathbb{RP}^2$ out of smaller substructures. Then, we give an overview of the probabilistic techniques required to locate these substructures.

2 Deconstructing $\mathbb{RP}^2$

Our proof of Theorem 1.1 begins by identifying conditions under which a 3-uniform hypergraph $\mathcal{H}$ contains a sub-hypergraph homeomorphic to $\mathbb{RP}^2$.

We decompose $\mathbb{RP}^2$ as two copies of $\mathbb{D}^2$ attached to $\mathbb{S}^1 \vee \mathbb{S}^1$. Consider the standard representation of $\mathbb{RP}^2$ as a disk with boundary glued to itself antipodally — this is pictured

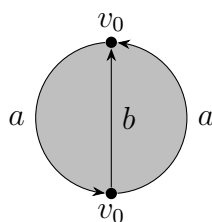


Figure 1: Two loops a and b in $\mathbb{RP}^2$ based at the point v_0 . Here, $\mathbb{RP}^2$ is depicted as a disk with boundary points identified antipodally.

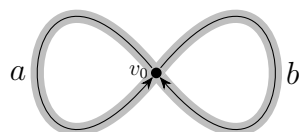


Figure 2: Two loops a and b in $\mathbb{S}^1 \vee \mathbb{S}^1$ sharing the same basepoint v_0 .

in Fig. 1. Let a and b be the loops in $\mathbb{RP}^2$ depicted, with a traversing half the boundary of the disk and b a diameter of the disk. The union of a and b , shown in Fig. 2, is a subspace of $\mathbb{RP}^2$ homeomorphic to $\mathbb{S}^1 \vee \mathbb{S}^1$. Moreover, $\mathbb{RP}^2$ can be recovered from this subspace by attaching two copies of $\mathbb{D}^2$ — one corresponding to each semicircular region of Fig. 1 — to the concatenated loops ab and $a^{-1}b$. This is summarized in the following proposition.

Proposition 2.1. *Let a and b be the two loops in $\mathbb{S}^1 \vee \mathbb{S}^1$ shown in Fig. 2. Form a CW complex from $\mathbb{S}^1 \vee \mathbb{S}^1$ by attaching one disk to the loop ab and another disk to the loop $a^{-1}b$. The resulting topological space is homeomorphic to $\mathbb{RP}^2$.*

Let $\mathbb{D}^{2-}$ be the quotient of $\mathbb{D}^2$ obtained by gluing together two points x, y on the boundary of $\mathbb{D}^2$. Proposition 2.1 decomposes $\mathbb{RP}^2$ as a union of two copies of $\mathbb{D}^{2-}$ intersecting on their shared boundary, a subspace of $\mathbb{RP}^2$ homeomorphic to $\mathbb{S}^1 \vee \mathbb{S}^1$.

Now, suppose $\mathcal{H}$ is a 3-uniform hypergraph. For a vertex $u \in V(\mathcal{H})$, we denote by $\mathcal{H}_u$ its *link graph*, the graph on $V(\mathcal{H}) \setminus \{u\}$ whose edges vw correspond to 3-edges $uvw \in E(\mathcal{H})$. For distinct vertices u and u' of $\mathcal{H}$, we write $\mathcal{H}_{u,u'} = \mathcal{H}_u \cap \mathcal{H}_{u'}$; that is, $\mathcal{H}_{u,u'}$ is the graph on $V(\mathcal{H}) \setminus \{u, u'\}$ with edge set $E(\mathcal{H}_u) \cap E(\mathcal{H}_{u'})$.

One might attempt to build $\mathbb{RP}^2$ using the following naïve approach. Choose vertices u, u' and cycles $C, C' \subseteq \mathcal{H}_{u,u'}$ so that C and C' intersect in a single vertex v_0 , implying that $C \cup C'$ is homeomorphic to $\mathbb{S}^1 \vee \mathbb{S}^1$. Let $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{H}$ be sub-hypergraphs induced by the edge sets $E(\mathcal{A}) = \{ue : e \in E(C) \cup E(C')\}$ and $E(\mathcal{A}') = \{u'e : e \in E(C) \cup E(C')\}$. One hopes that $\mathcal{A}$ and $\mathcal{A}'$ are copies of $\mathbb{D}^{2-}$ whose union is homeomorphic to $\mathbb{RP}^2$, and indeed this is almost true. However, the 1-simplex uv_0 (resp. $u'v_0$) is contained in four different edges of $\mathcal{A}$ (resp. $\mathcal{A}'$), so neither $\mathcal{A}$ nor $\mathcal{A}'$ is homeomorphic to $\mathbb{D}^{2-}$.

To obtain a homeomorphic copy of $\mathbb{RP}^2$, we alter the hypergraphs $\mathcal{A}$ and $\mathcal{A}'$ to avoid these four-way intersections. The resulting construction is pictured in Fig. 3. Let v_1, v_2

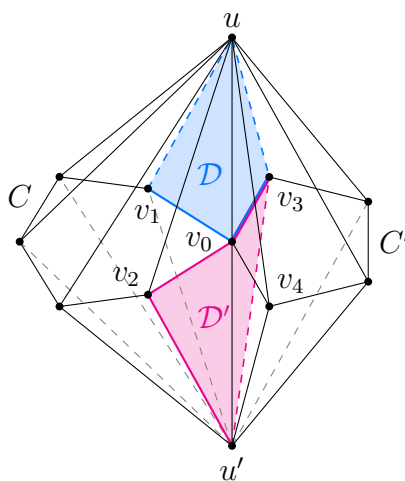


Figure 3: Building $\mathbb{RP}^2$ from cycles $C, C' \subseteq \mathcal{H}_{u,u'}$ and disks $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{H}$.

be the two neighbors of v_0 in C , and let v_3, v_4 be the two neighbors of v_0 in C' . Consider the edge subsets $D = \{uv_0v_1, uv_0v_3\} \subseteq E(\mathcal{A})$ and $D' = \{u'v_0v_2, u'v_0v_4\} \subseteq E(\mathcal{A}')$, which correspond to disks with boundaries $v_0v_1uv_3v_0$ and $v_0v_2u'v_4v_0$ respectively. We locate alternate sub-hypergraphs $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{H}$ homeomorphic to disks with the same boundaries, and replace D and D' with them. If $\mathcal{D}$ and $\mathcal{D}'$ are chosen appropriately, the altered hypergraphs $(\mathcal{A} \setminus D) \cup \mathcal{D}$ and $(\mathcal{A}' \setminus D') \cup \mathcal{D}'$ are homeomorphic to $\mathbb{D}^{2-}$ with shared boundary $C \cup C'$. In fact, they are created by attaching disks to $C \cup C'$ along the loops $v_0v_2 \overset{C}{\cdots} v_1v_0v_3 \overset{C'}{\cdots} v_4v_0$ and $v_0v_1 \overset{C}{\cdots} v_2v_0v_3 \overset{C'}{\cdots} v_4v_0$, respectively. Using Proposition 2.1, one can show that their union is homeomorphic to $\mathbb{RP}^2$.

3 Probabilistic Techniques

We have reduced Theorem 1.1 to finding substructures $C, C', \mathcal{D}, \mathcal{D}'$ of a 3-uniform hypergraph $\mathcal{H}$ arranged as in Fig. 3. We locate these substructures via a probabilistic approach, analyzing the likelihood that a randomly chosen subset of $V(\mathcal{H})$ will contain each of these substructures. To quantify these probabilities, we require some new definitions. Write $U \subseteq_p V$ to indicate that U is a randomly chosen subset of V , containing each vertex independently with probability p .

Definition 3.1. Fix $p, \epsilon \in (0, 1]$. Let $\mathcal{H}$ be a 3-uniform hypergraph and let $x_1, \dots, x_4$ be four distinct vertices of $\mathcal{H}$. Sampling $U \subseteq_p V(\mathcal{H})$, let $A_{x_1x_2x_3x_4}$ be the event that there is some sub-hypergraph $\mathcal{D} \subseteq \mathcal{H}[\{x_1, \dots, x_4\} \cup U]$ which is homeomorphic to a disk bounded by the 4-cycle $x_1x_2x_3x_4$, and which contains neither 1-simplex x_1x_3 or x_2x_4 . We say the 4-cycle $x_1 \cdots x_4$ is (p, ϵ) -disk-coverable if $\Pr[A_{x_1 \cdots x_4}] \geq 1 - \epsilon$.

Kupavskii, Polyanskii, Tomon, and Zakharov implicitly studied disk-coverability when upper-bounding the Turán number of the torus in [5]. They introduced the following

related notion.

Definition 3.2. Fix $p, \epsilon \in (0, 1]$. Let G be a graph and $e = xy$ an edge of G . Sample $U \subseteq_p V(G)$ and let A_e be the event that there is a cycle containing xy in $G[U \cup \{x, y\}]$. We say the edge e is (p, ϵ) -admissible if $\Pr[A_e] \geq 1 - \epsilon$.

The concept of admissibility is useful due to the following observation: if an edge vw in a link graph $\mathcal{H}_{u,u'}$ is (p, ϵ) -admissible, then the 4-cycle $uvu'w$ is (p, ϵ) -disk-coverable. This is because any cycle $v_0 \cdots v_\ell$ with $v = v_0$ and $w = v_\ell$ gives rise to a sub-hypergraph with edge set

$$\bigcup_{i=0}^{\ell-1} \{v_i v_{i+1} u, v_i v_{i+1} u'\},$$

which is homeomorphic to a disk with boundary $uvu'w$.

At this point, we may sketch the proof of Theorem 1.1. Given a hypergraph $\mathcal{H}$, we locate vertices $u, u', v_0, \dots, v_4$, cycles $C, C' \subseteq \mathcal{H}_{u,u'}$, and disks $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{H}$ as pictured in Fig. 3.

Proof Overview of Theorem 1.1. Let $p = 1/4$ and fix ϵ (to be determined later). Let $\mathcal{H}$ be a 3-uniform hypergraph with at least $cn^{5/2}$ edges. If c is sufficiently large in terms of p and ϵ then, using techniques from [5], we may pass to a sub-hypergraph $\mathcal{H}' \subseteq \mathcal{H}$ with at least $\frac{\epsilon}{2}n^{5/2}$ edges such that for any neighboring edges $xyz, x'yz \in \mathcal{H}'$, the 4-cycle $xyx'z$ is (p, ϵ) -disk-coverable in $\mathcal{H}$.

We locate vertices u, u' , a graph $G \subseteq \mathcal{H}'_{u,u'}$, and incident edges $v_0v_1, v_0v_3 \in E(G)$ which are both (p, δ) -admissible in G , with $\delta = 1/3$. Additionally, we show that $\deg_G(v_0)$ is at most some fixed constant d , which is computed in terms of the admissibility parameters (p, δ) . The details of this step may be found in [10].

Choose $v_2 \in N_G(v_0)$ uniformly at random and partition $V(G) = U_1 \cup \dots \cup U_4$ by placing each vertex in a given set U_i independently with probability $p = 1/4$. Consider the following three events.

- (A1) There are cycles C, C' satisfying the inclusions $v_0v_1 \subset C \subset G[U_1 \cup \{v_0, v_1\}]$ and $v_0v_3 \subset C' \subset G[U_2 \cup \{v_0, v_3\}]$.
- (A2) The cycle C described in (A1) contains $v_1v_0v_2$ as a subpath.
- (A3) There are $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{H}$ homeomorphic to disks with boundaries $uv_1v_0v_3$ and $u'v_2v_0v_3$ whose non-boundary vertices are contained in U_3 and U_4 , respectively. Moreover, $\mathcal{D}$ does not contain the 1-simplex uv_0 , and $\mathcal{D}'$ does not contain the 1-simplex $u'v_0$.

If all three events hold simultaneously, then the structures $C, C', \mathcal{D}, \mathcal{D}'$ do not intersect except at the vertices u, u', v_0, v_1, v_3 . To obtain a homeomorphic copy of $\mathbb{RP}^2$, we must additionally ensure that the structures do not contain any of these five vertices unless mentioned in (A1) and (A3). This is summarized in the following two conditions.

- (B1) We have $v_3 \notin C$ and $v_1 \notin C'$.

(B2) We have $u' \notin \mathcal{D}$ and $u \notin \mathcal{D}'$.

It remains to show that the five events (A1), (A2), (A3), (B1), (B2) occur simultaneously with positive probability.

Because the edges v_0v_1 and v_0v_3 are (p, δ) -admissible in G , the event (A1) occurs with probability at least $1 - 2\delta = 1/3$. To additionally show that $v_3 \notin C$ and $v_1 \notin C'$, as in (B1), we check that the edges v_0v_1 and v_0v_3 are admissible (with suitable parameters) in $G - v_3$ and $G - v_1$, respectively. If $xy \in E(G)$ is a (p, δ) -admissible edge in G and $G' = G - z$ is a subgraph created by deleting a third vertex z from G , then

$$\begin{aligned} & \Pr_{U' \subseteq_p V(G')} [\nexists \text{ cycle in } G'[U' \cup \{x, y\}] \text{ containing } xy] \\ &= \Pr_{U \subseteq_p V(G)} [\nexists \text{ cycle in } G[U \cup \{x, y\}] \text{ containing } xy \mid z \notin U] \\ &\leq \frac{\Pr_{U \subseteq_p V(G)} [\nexists \text{ cycle in } G[U \cup \{x, y\}] \text{ containing } xy]}{\Pr_{U \subseteq V(G)} [z \notin U]} \leq \frac{\delta}{1-p}. \end{aligned}$$

It follows that the edges v_0v_1 and v_0v_3 are $(p, \frac{\delta}{1-p})$ -admissible in $G - v_3$ and $G - v_1$, respectively. Thus, with probability at least $1 - \frac{2\delta}{1-p} = 1/9$, there are cycles C, C' satisfying (A1) and (B1).

Notice that (A2) is independent of (A1) and (B1) — the latter two events depend only on the choice of $U_1, \dots, U_4$, while (A2) depends on the choice of v_2 . It follows that (A1), (B1), (A2) simultaneously hold with probability at least $\frac{1}{9} \Pr[(A2)] = 1/9d$.

Lastly, we consider (A3) and (B2). Observe that uv_1v_0 and uv_3v_0 are neighboring edges of $\mathcal{H}'$, so $uv_1v_0v_3$ is (p, ϵ) -disk-coverable in $\mathcal{H}$. Similarly, $u'v_2v_0v_3$ is also (p, ϵ) -disk-coverable in $\mathcal{H}$. It follows that (A3) holds with probability at least $1 - 2\epsilon$. To additionally include (B2), we note that $uv_1v_0v_3$ and $u'v_2v_0v_3$ are $(p, \frac{\epsilon}{1-p})$ -disk-coverable in $\mathcal{H} - u'$ and $\mathcal{H} - u$, respectively, by a calculation analogous to that for (B1) above. Thus, (A3) and (B2) hold simultaneously with probability at least $1 - \frac{2\epsilon}{1-p} \geq 1 - 3\epsilon$.

By a union bound, the five events (A1), (B1), (A2), (A3), (B2) hold simultaneously with probability at least $1/9d - 3\epsilon$. Thus, assuming that ϵ was chosen to satisfy $\epsilon < 1/27d$, there is a sub-hypergraph of $\mathcal{H}$ homeomorphic to $\mathbb{RP}^2$. $\square$

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