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The Rado Multiplicity Problem in Vector Spaces over Finite Fields

(Extended abstract)

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Abstract

We study an analogue of the Ramsey multiplicity problem for additive structures, establishing the minimum number of monochromatic 3-APs in 3-colorings of $\mathbb{F}_3^n$ and obtaining the first non-trivial lower bound for the minimum number of monochromatic 4-APs in 2-colorings of $\mathbb{F}_5^n$. The former parallels results by Cumings et al. [3] in extremal graph theory and the latter improves upon results of Saad and Wolf [13]. Lower bounds are notably obtained by extending the flag algebra calculus of Razborov [11].

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1 Introduction

In 1959 Goodman [7] proved that asymptotically at least a quarter of all vertex triples in any graph must either form a clique or an independent set. This lead to the study of the *Ramsey multiplicity problem*, where one would like to determine the minimum number of monochromatic cliques of prescribed size over any edge-coloring of the complete graph [5,

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15, 2]. Recently there has been an increased interest in studying the arithmetic analogue of this type of question, originally initiated when Graham, Rödl, and Ruczinsky [8] gave an asymptotic lower bound for the minimum number of monochromatic Schur triples in 2-colorings of the first n integers in 1996, see also [12, 14, 4, 1].

In this extended abstract, we focus on the analogue of the Ramsey multiplicity problem for specific additive structures in vector spaces over finite fields of small order. Let $q \in \mathbb{N}$ be a fixed prime power throughout and write $\mathbb{F}_q$ for the finite field with q elements. Given a subset $T \subseteq \mathbb{F}_q^n$ and a linear map L defined by some matrix $A \in \mathcal{M}^{r\times m}(\mathbb{Z})$ with integer entries co-prime to q, we are interested in studying the set $\mathcal{S}_L(T) = \{\mathbf{s} = (s_1, \ldots, s_m) \in T^m : L(\mathbf{s}) = \mathbf{0} \text{ and } s_i \neq s_j \text{ for } i \neq j\}$ of solutions with all-distinct entries in T. Throughout, we will assume that A is of full rank and that $\mathcal{S}_L(\mathbb{F}_q^n) \neq \emptyset$. We will also write $s_L(T) = |\mathcal{S}_L(T)|/|\mathcal{S}_L(\mathbb{F}_q^n)|$. Writing $[c] = \{1, \ldots, c\}$ for some given number of colors $c \in \mathbb{N}$, we call $\gamma : \mathbb{F}_q^n \to [c]$ a *c-coloring* of dimension $\dim(\gamma) = n$ and let $\gamma^{(i)}$ denote the set of elements colored with color $1 \leq i \leq c$ as well as $\Gamma_c(q, n)$ for the set of all c-colorings of $\mathbb{F}_q^n$. The *Rado multiplicity problem* is concerned with determining

$$m_c(L, q) = \lim_{n\to\infty} \min_{\gamma\in\Gamma_c(q,n)} s_L(\gamma^{(1)}) + \ldots + s_L(\gamma^{(c)}). \tag{1}$$

The limit exists by monotonicity and we have $0 \leq m_c(L, q) \leq 1$ by definition. Rado's theorem establishes that $m_c(L, q) > 0$ and we say that L is *c-common* for q if $m_c(L, q) = c^{1-m}$, that is if the minimum number of monochromatic solutions is attained in expectation by a uniform random coloring. For $r = 1$ a result of Cameron, Cilleruelo, and Serra [1] establishes that any L is 2-common if m is odd. When m is even, Saad and Wolf [13] showed that any L where the coefficients can be partitioned into pairs, with each pair summing to zero, is 2-common. Fox, Pham, and Zhao [6] showed that this sufficient condition is in fact also necessary. The case when $r > 1$ is much less understood, with Kamčev, Liebenau, and Morrison [9] recently characterizing a large family of non-common linear maps by showing that any L that 'induces' some smaller 2×4 linear map is uncommon. Focusing on specific values of q, Král, Lamaison, and Pach [10] also recently characterized the 2-common L for $q = 2$ when $r = 2$ and m is assumed to be odd. When $q = 5$, the most relevant additive structures to study is that of 4-APs. Saad and Wolf [13] showed that they are not 2-common by establishing an upper bound of $1/8 - 7 \cdot 2^{10} \cdot 5^{-2} \approx 0.1247 < 2^{-4}$. We establish the first non-trivial lower bound for this problem and an improved upper bound.

Proposition 1.1. *We have* $1/10 < m(L_{4\text{-}AP}, 5) \leq 13/126 = 0.1\overline{031746}$.

Going beyond 4-APs, we can also show that $m(L_{5\text{-AP}}, 5) \leq 1/26 < 2^{-4}$, establishing that 5-APs are likewise not 2-common in $\mathbb{F}_5$, but in this case did not obtain any meaningful lower bound. The study of monochromatic structures in colorings with more than two colors has also proven relevant in extremal graph theory. Most notably, Cummings et al. [3] extended the results of Goodman [7] by establishing the exact Ramsey multiplicity of triangles in 3-colorings and showing that they are not 3-common despite being 2-common. We consider a similar question and establish the exact multiplicity of 3-APs in 3-colorings of $\mathbb{F}_3^n$.

Theorem 1.2. *We have* $m_3(L_{3\text{-}AP}, 3) = 1/27$.

We can also show that $0.04486 \leq m_3(L_{\text{Schur}}, 2) \leq 1/16$ as well as $m_3(L_{\text{Schur}}, 3) \leq 7/81$, establishing that Schur triples are also not 3-common for $q = 2$ and $q = 3$. Upper bounds of all results are obtained through explicit blowup-type constructions. Lower bounds in the graph theoretic setting have recently been obtained through a computational approach relying on flag algebras due to Razborov [11]. This approach has been extended to different contexts, but so far seems to not have been explored in the arithmetic setting.

2 The correct notion of isomorphism

Let us omit q and c from notation, so in particular we write $\Gamma(n) = \Gamma_c(q, n)$ for the set of all c-colorings of dimension n as well as $\Gamma = \bigcup_{n=0}^{\infty} \Gamma(n)$. The 0-dimensional vector space consist of a single point, that is $\mathbb{F}_q^0 = \{0\}$, and we write e_j for the j-th canonical unit basis vector of $\mathbb{F}_q^n$ for $1 \leq j \leq n$ as well as e_0 for the zero vector.

Definition 2.1. *We refer to an affine linear map* $\varphi : \mathbb{F}_q^k \to \mathbb{F}_q^n$ *as a* morphism *and say that it is* t-fixed *for some* $t \geq 0$ *if* $\varphi(e_j) = e_j$ *for all* $0 \leq j \leq t$. *A morphism is a* monomorphism *whenever it is injective and a monomorphism is an* isomorphism *whenever* $n = k$.

Out of notational convenience, we extend the range of t to -1 in order to include *unfixed* morphisms and will always use t^+ to denote $\max\{t, 0\}$. For a given $t \geq -1$ and $n \geq k \geq t^+$, we let $\mathrm{M}_t(k; n)$ denote the set of t-fixed morphisms from $\mathbb{F}_q^k$ to $\mathbb{F}_q^n$ up to t-fixed isomorphism of $\mathbb{F}_q^k$. We likewise write $\mathrm{Mon}_t(k; n)$ for the set of monomorphisms with the same properties. Given $k_1, \ldots, k_m \geq t^+$ and $n \geq k_1 + \ldots + k_m - (m-1)\, t^+$, we let $\mathrm{Mon}_t(k_1, \ldots, k_m; n)$ denote the set of all tuples of monomorphisms $(\varphi_1, \ldots, \varphi_m) \in \mathrm{Mon}_t(k_1; n) \times \ldots \times \mathrm{Mon}_t(k_m; n)$ overlapping only in the t-fixed subspace.

Using these notions, we say two colorings $\gamma_1, \gamma_2 \in \Gamma(n)$ are *t-fixed isomorphic* for some $t \geq -1$, denoted by $\gamma_1 \cong_t \gamma_2$, if there exists a t-fixed isomorphism $\varphi : \mathbb{F}_q^n \to \mathbb{F}_q^n$ satisfying $\gamma_1 \equiv \gamma_2 \circ \varphi$. We let $\Gamma_t(n) = \Gamma(n)/\cong_t$ denote the set of all c-colorings of $\mathbb{F}_q^n$ up to t-fixed isomorphism and also write $\Gamma_t = \bigcup_{n \geq t^+} \Gamma_t(n)$. Given $k_1, \ldots, k_m \geq t^+$ and $n \geq k_1 + \ldots + k_m - (m-1)\, t^+$, the *density* $p_t(\delta_1, \ldots, \delta_m; \gamma)$ of some colorings $\delta_1 \in \Gamma_t(k_1)$, ..., $\delta_m \in \Gamma_t(k_m)$ in $\gamma \in \Gamma_t(n)$ is defined as the probability that a a tuple of t-fixed monomorphism chosen uniformly at random from $\mathrm{Mon}_t(k_1, \ldots, k_m; n)$ induces copies of $\delta_1, \ldots, \delta_m$ in γ. For $n \geq k \geq t^+$, we also let the *degenerate density* $p_t^d(\delta; \gamma)$ of some $\delta \in \Gamma_t(k)$ in γ denote the probability that a not-necessarily-injective t-fixed morphism does the same.

3 The correct notion of solution

In order to develop the flag algebra approach, the density of solutions needs to be representable as the weighted density of particular colorings, motivating the following definition.

Definition 3.1. *For any $t \geq -1$ and $n \geq t^+$, the* t-fixed dimension $\dim_t(s)$ *of* $\mathbf{s} \in \mathcal{S}_L(\mathbb{F}_q^n)$ *is the smallest $k \geq t^+$ for which there exists a t-fixed k-dim. subspace of $\mathbb{F}_q^n$ containing* $\mathbf{s}$.

We will only need the unfixed and 0-fixed dimension and denote by $\dim_t(L)$ the largest t-fixed dimension of any solution to a given linear map L. In general, $\dim_t(L) = m - r + t$ for any linear map L when $t \geq 0$ as well as $\dim_{-1}(L) = m - r - 1$ when L is *invariant*, that is if for any solution $\mathbf{s} = (x_1, \dots, x_m) \in \mathcal{S}_L(\mathbb{F}_q^n)$ and element $a \in \mathbb{F}_q^n$ we have $a + \mathbf{s} = (a + x_1, \dots, a + x_m) \in \mathcal{S}'_L(\mathbb{F}_q^n)$. We say that L is *admissible* if $t \geq 0$ or if $t = -1$ and L is invariant. A solution $\mathbf{s} \in \mathcal{S}_L(\mathbb{F}_q^n)$ for some admissible L is *t-fixed fully dimensional* if $\dim_t(s)$ attains the respective upper bound. For a given set $T \subseteq \mathbb{F}_q^n$, we denote the set of fully dimensional solutions to some admissible L by $\mathcal{S}_L^t(T)$ and write $s_L^t(T) = |\mathcal{S}_L^t(T)| \,/\, |\mathcal{S}_L^t(\mathbb{F}_q^n)|$.

The important property that we make use of is that each fully-dimensional solution defines a unique $\dim(L)$-dimensional t-fixed subspace in which it lies and that for any $t \geq -1$, admissible L, and $n \geq t \geq 0$, the number of solutions in a subset of $\mathbb{F}_q^n$ is invariant under t-fixed isomorphism. The same would not hold for $t = -1$ if L was not invariant.

4 The flag algebras for additive structures

For any $t \geq 0$, we refer to elements of $\Gamma_t(t) = \Gamma(t)$ as *types* of dimension t. We also introduce a unique *empty type*, denoted by $\varnothing$, of dimension $t = -1$. For a given type τ of dimension t, we refer to a coloring $F \in \Gamma_t(n)$ satisfying $F \circ \mathrm{id}_{t,n} \equiv \tau$ as a *flag of type* τ, where $\mathrm{id}_{t,n}$ denotes the unique t-fixed isomorphism from $\mathbb{F}_q^t$ to $\mathbb{F}_q^n$ and the requirement is vacantly true for $t = -1$. We will write $\mathcal{F}_n^\tau$ for the set of all flags of given type τ and dimension n as well as $\mathcal{F}^\tau = \bigcup_n \mathcal{F}_n^\tau$.

Definition 4.1. *The* flag algebra $\mathcal{A}^\tau$ *of type τ is given by equipping $\mathbb{R}\mathcal{F}^\tau/\mathcal{K}^\tau$, where $\mathcal{K}^\tau = \{F - \sum_{F' \in \mathcal{F}_n^\tau} p_t(F; F')\, F' : F \in \mathcal{F}^\tau, n \geq \dim(F)\}$, with the product given by the the bilinear extension of $F_1 \cdot F_2 = \sum_{H \in \mathcal{F}_n^\tau} p_t(F_1, F_2; H)\, H + \mathcal{K}^\tau$ defined for any two flags $F_1, F_2 \in \mathcal{F}^\tau$ and arbitrary $n \geq \dim(F_1) + \dim(F_2) - \dim(\tau)$.*

Assume we are given a parameter $\lambda : \Gamma \to \mathbb{R}$ that is invariant under t_λ-fixed isomorphisms for some $t_\lambda \geq -1$ and that satisfies $\lambda(\gamma) = \sum_{\beta \in \Gamma_{t_\lambda}(n)} \lambda(\beta)\, p_{t_\lambda}(\beta, \gamma)$ for some $n_\lambda \in \mathbb{N}$ and all $\gamma \in \Gamma_{t_\lambda}$, where $n_\lambda \leq n \leq \dim(\gamma)$. Monochromatic fully-dimensional solutions to a given linear map L define such a parameter with $t_\lambda = 0$ for general L and $t_\lambda = -1$ for invariant ones, where in either case $n_\lambda = \dim_{t_\lambda}(L)$. We are interested in determining

$$\lambda^\star = \lim_{n \to \infty} \min_{\gamma \in \Gamma_{t_\lambda}(n)} \lambda(\gamma). \tag{2}$$

Writing $C_\lambda^\tau = \sum_{\beta \in \mathcal{F}_{n_\lambda}^\tau} \lambda(\beta)\, \beta$ for any type τ of dimension t_λ, our problem of determining $\lambda^\star$ can be restated through the conic optimization problem

$$\lambda^\star = \max\{\lambda' \in \mathbb{R} : C_\lambda^\tau \geq \lambda' \text{ for all types } \tau \text{ of dimension } t_\lambda\}, \tag{3}$$

where we write $\mathrm{Hom}^+(\mathcal{A}^\tau, \mathbb{R})$ for the set of positive homomorphisms, that is algebra homomorphisms $\phi \in \mathrm{Hom}(\mathcal{A}^\tau, \mathbb{R})$ satisfying $\phi(F) \geq 0$ for any $F \in \mathcal{F}^\tau$, and $\mathcal{S}^\tau = \{f \in \mathcal{A}^\tau : \phi(f) \geq 0 \text{ for all } \phi \in \mathrm{Hom}^+(\mathcal{A}^\tau, \mathbb{R})\}$ for the *semantic cone* of type τ. Noting that we can define a linear *downward operator* $[\![\cdot]\!]_{t_\lambda} \colon \mathcal{A}^\tau \to \mathcal{A}^{\tau_\lambda}$ for any type τ of dimension $t \geq t_\lambda$ that satisfies $[\![\mathcal{S}^\tau]\!]_{t_\lambda} \subseteq \mathcal{S}^{\tau_\lambda}$, we can derive a lower bound by defining a set of types $\mathcal{T}$ as well as sets of algebra elements $\mathcal{B}_{\tau'} \subset \mathcal{A}^{\tau'}$ and establishing that

$$C_\lambda^\tau \geq \lambda' + \sum_{\tau' \in \mathcal{T}} \sum_{f \in \mathcal{B}_{\tau'}} [\![f^2]\!]_{t_\lambda}. \tag{4}$$

References

[1] P. J. Cameron, J. Cilleruelo, and O. Serra. On monochromatic solutions of equations in groups. *Revista Matemática Iberoamericana*, 23(1):385–395, 2007.

[2] D. Conlon. On the Ramsey multiplicity of complete graphs. *arXiv preprint arXiv:0711.4999*, 2007.

[3] J. Cummings, D. Král', F. Pfender, K. Sperfeld, A. Treglown, and M. Young. Monochromatic triangles in three-coloured graphs. *Journal of Combinatorial Theory, Series B*, 103(4):489–503, 2013.

[4] B. A. Datskovsky. On the number of monochromatic Schur triples. *Advances in Applied Mathematics*, 31(1):193–198, 2003.

[5] P. Erdős. On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 7(3):459–464, 1962.

[6] J. Fox, H. T. Pham, and Y. Zhao. Common and Sidorenko linear equations. *The Quarterly Journal of Mathematics*, 72(4):1223–1234, 2021.

[7] A. W. Goodman. On sets of acquaintances and strangers at any party. *The American Mathematical Monthly*, 66(9):778–783, 1959.

[8] R. Graham, V. Rödl, and A. Ruciński. On Schur properties of random subsets of integers. *Journal of Number Theory*, 61(2):388–408, 1996.

[9] N. Kamčev, A. Liebenau, and N. Morrison. Towards a characterisation of Sidorenko systems. *arXiv preprint arXiv:2107.14413*, 2021.

[10] A. Lamaison, P. P. Pach, et al. Common systems of two equations over the binary field. In *Discrete Mathematics Days 2022*, pages 169–173. Editorial de la Universidad de Cantabria, 2022.

[11] A. A. Razborov. Flag algebras. *The Journal of Symbolic Logic*, 72(4):1239–1282, 2007.

[12] A. Robertson and D. Zeilberger. A 2-coloring of $[1, n]$ can have $(1/22)n^2 + o(n)$ monochromatic Schur triples, but not less! *The Electronic Journal of Combinatorics*, 5(1):R19, 1998.

[13] A. Saad and J. Wolf. Ramsey multiplicity of linear patterns in certain finite abelian groups. *The Quarterly Journal of Mathematics*, 68(1):125–140, 2017.

[14] T. Schoen. The number of monochromatic Schur triples. *European Journal of Combinatorics*, 20(8):855–866, 1999.

[15] A. Thomason. A disproof of a conjecture of Erdős in Ramsey theory. *Journal of the London Mathematical Society*, 2(2):246–255, 1989.