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On Hypergraph Supports

(Extended abstract)

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Abstract

Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph. A support is a graph Q on X such that for each $E \in \mathcal{E}$, the subgraph of Q on the elements in E is connected. We consider hypergraphs defined on a host graph. Given a graph $G = (V, E)$, with $c : V \to \{\mathbf{r}, \mathbf{b}\}$ and a collection of connected subgraphs $\mathcal{H}$ of G, a primal support is a graph Q on $\mathbf{b}(V)$ such that for each $H \in \mathcal{H}$, the subgraph $Q[\mathbf{b}(H)]$ on vertices $\mathbf{b}(H) = H \cap c^{-1}(\mathbf{b})$ is connected. A *dual support* is a graph Q^* on $\mathcal{H}$ s.t. for each $v \in X$, the subgraph $Q^*[\mathcal{H}_v]$ is connected, where $\mathcal{H}_v = \{H \in \mathcal{H} : v \in H\}$. We present sufficient conditions on the host graph and hyperedges so that the resulting support comes from a restricted family.

We primarily study two classes of graphs: (1) If the host graph has genus g and the hypergraphs satisfy a topological condition of being *cross-free*, then there is a primal and a dual support of genus at most g. (2) If the host graph has treewidth t and the hyperedges satisfy a combinatorial condition of being *non-piercing*, then there exist primal and dual supports of treewidth $O(2^t)$. We show that this exponential blow-up is sometimes necessary. As an intermediate case, we also study the case when the host graph is outerplanar. Finally, we show applications of our results to packing and covering, and coloring problems on geometric hypergraphs.

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1 Introduction

A hypergraph $(X, \mathcal{E})$ is defined by a set X of elements and a collection $\mathcal{E}$ of subsets of X. A *support* is a graph Q on X s.t. $\forall\ E \in \mathcal{E}$, the subgraph $Q[E]$ induced by the vertices of E is connected. The notion of a support was introduced by Voloshina and Feinberg [28] in the context of VLSI circuits. Since then, this notion has found wide applicability in several areas, such as visualizing hypergraphs [7, 8, 9, 10, 11, 16, 18], in the design of networks [2, 4, 5, 12, 17, 22, 25], and similar notions have been used in the analysis of local search algorithms for geometric problems [3, 6, 13, 23, 24, 27].

Any hypergraph clearly has a support: a complete graph on X is a support. The problem becomes interesting if we introduce a global constraint on the graph that is in *tension* with the *local* connectivity requirement for each hyperedge. In particular, we are interested in restrictions on the hypergraph that guarantees the existence of a support from a *sparse* family of graphs, namely a family with sub-linear sized separators[1]. A support from a family $\mathcal{G}$ of graphs is called a $\mathcal{G}$ support.

Our motivation to study the existence of such supports comes primarily from the analysis of *local search* algorithms for several packing and covering problems defined by geometric objects in the Euclidean plane. With the aim of extending the analysis techniques from the plane to other surfaces, we study hypergraphs defined on a sparse *host graph.*

A *geometric hypergraph* is defined by a set P of points in $\mathbb{R}^2$, and a set $\mathcal{D}$ of *regions*, or subsets of $\mathbb{R}^2$, where the hyperedges are defined by $D \cap P$ for each $D \in \mathcal{D}$. We call this hypergraph a *primal hypergraph* to distinguish it from other hypergraphs we will define shortly.

If $\mathcal{D}$ is a collection of *pseudodisks*,[2] Pyrga and Ray [26] showed that the hypergraph $(P, \mathcal{D})$ admits a planar support. Raman and Ray in [27] generalized this result to show that if the regions in a geometric hypergraph are *non-piercing*[3], then the hypergraph $(P, \mathcal{D})$ admits a planar support. The authors also show that the *dual hypergraph* $(\mathcal{D}, \{\mathcal{D}_p\}_{p \in P})$, where for each $p \in P$, $\mathcal{D}_p = \{D \in \mathcal{D} : D \ni p\}$ admits a planar support.

For an arrangement $\mathcal{D}$ of pseudodisks in the plane in general position[4], let G denote the *dual arrangement graph*, whose vertices are the cells in the arrangement, and two cells are adjacent if they share an arc of the boundary of a pseudodisk. It is easy to see that the dual arrangement graph G is a plane graph, and each pseudodisk D corresponds to a connected subgraph H_D of G. Further, for any pair of subgraphs $H_D, H_{D'}$ corresponding to

[1]A graph has a sublinear sized separator if there is some constant $\epsilon > 0$, and $c > 0$ such that there is a set S of size $O(|V|^{1-\epsilon})$ such that $G \setminus S$ can be partitioned into two subgraphs A and B s.t. there is no edge in $G \setminus S$ between a vertex in A and a vertex in B, and s.t. $|V(A)|, |V(B)| \leq c|V(G)|$.

[2]A collection of simple Jordan curves define a set of pseudocircles if each pair intersects either 0 or twice. A collection of bounded regions whose boundaries are a collection pseudocircles are a collection of pseudodisks.

[3]a collection of regions $\mathcal{H}$, where each $H \in \mathcal{H}$ is a path-connected region bounded by a simple Jordan curve (possibly with holes) is non-piercing if both $H \setminus H'$ and $H' \setminus H$ are connected.

[4]An arrangement $\mathcal{D}$ of pseudodisks in the plane is in general position if there are no three pseudodisks whose boundaries pass through a common point, and at each intersection point, the boundaries of the pair of pseudodisks defining the intersection point properly cross.

pseudodisks D and D', respectively, the graphs $G[H_D \setminus H_{D'}]$ and $G[H_{D'} \setminus H_D]$ are connected, i.e., $\mathcal{H} = \{H_D : D \in \mathcal{D}\}$ is a collection of *non-piercing* subgraphs of G. This motivates the following definition:

Definition 1 (Non-piercing)**.** *For a graph G, a collection of connected subgraphs $\mathcal{H}$ of G is non-piercing if for any two subgraphs $H, H' \in \mathcal{H}$, both $G[H \setminus H']$ and $G[H' \setminus H]$ are connected.*

For each cell c in the arrangement $\mathcal{D}$, let v_c denote the vertex in the dual arrangement graph G corresponding to this cell. If $c \cap P \neq \emptyset$, set $color(v_c) = \mathbf{b}$, and if $c \cap P = \emptyset$, set $color(v_c) = \mathbf{r}$. Let $\mathbf{b}(V)$ and $\mathbf{r}(V)$ denote respectively, the subsets of vertices in $color^{-1}(\mathbf{b})$ and $color^{-1}(\mathbf{r})$. The result of Pyrga and Ray [26] translated in this context says that for a plane graph G with $color : V \to \{\mathbf{r}, \mathbf{b}\}$, and a collection of connected subgraphs $\mathcal{H}$ s.t. $\mathcal{H}$ is non-piercing, there is a planar support on $color^{-1}(\mathbf{b})$, i.e., a planar graph Q on the vertices $color^{-1}(\mathbf{b})$ s.t. the subgraph $Q[H]$ is connected for each $H \in \mathcal{H}$.

Our goal is to generalize the results above to the setting where the host graph comes from a non-trivial graph class. If the host graph G has bounded genus and $\mathcal{H}$ is a collection of connected non-piercing subgraphs of G, we may expect, as in the case of pseudodisk hypergraphs above, that there exist bounded genus supports for the primal and dual hypergraphs. While the statement may be true for primal hypergraphs. However, for the dual hypergraph, this is not true: Let G be the torus grid graph $T_{n,n} = C_n \square C_n$[29]. The subgraphs are the n non-contractible cycles perpendicular to the hole, and the n non-contractible cycles parallel to the hole. Each vertex of the graph belongs to exactly two subgraphs forcing them to be adjacent in the dual support, and thus the dual support is $K_{n,n}$ which is not embeddable on the torus for large enough n.

For bounded genus graphs, we show that if the subgraphs satisfy a condition of being *cross-free*, then there exists bounded genus supports for the primal as well as the dual hypergraphs. In the plane, the cross-free condition is weaker than the non-piercing condition, but these two conditions are incomparable on higher genus surfaces.

If we restrict attention to host graphs of bounded treewidth, we show that if the subgraphs are non-piercing then both the primal and dual supports have bounded treewidth. However, the treewidth of the support could be exponentially larger than the treewidth of the host graph. Along the way, we also consider outerplanar graphs, which have treewidth 2. Here, we show a distinction between the primal and dual settings. For the primal setting, the cross-free condition on the hypergraphs is sufficient to obtain an outerplanar support, while in the dual setting, we show that restricting the subgraphs to be non-piercing is a sufficient condition to obtain a dual outerplanar support.

2 Preliminaries

Let $\mathcal{H}$ be a collection of connected subgraphs of a graph $G = (V, E)$. This defines a hypergraph $(V, \mathcal{H})$. We call the pair $(G, \mathcal{H})$ a graph system. If G comes from a class $\mathcal{G}$ of graphs, and $\mathcal{H}$ satisfies property P we say that $(G, \mathcal{H})$ is a P-$\mathcal{G}$ system. Further, if G has genus g, we say that $(G, \mathcal{H})$ has genus g. In particular, if G is planar, we say that $(G, \mathcal{H})$ is a planar system. Let $c : V \to \{\mathbf{r}, \mathbf{b}\}$ be a coloring of V with two colors. Let $\mathbf{b}(V)$ and $\mathbf{r}(V)$ denote respectively $c^{-1}(\mathbf{b})$ and $c^{-1}(\mathbf{r})$.

For a graph system $(G, \mathcal{H})$, a *primal support* is a graph Q on $\mathbf{b}(V)$ s.t. $\forall\ H \in \mathcal{H}$, $Q[\mathbf{b}(H)]$ is connected[5], i.e., a support for the *primal hypergraph* $(V, \mathcal{H})$. A *dual support* is a graph Q^* on $\mathcal{H}$ s.t. $\forall\ v \in V$, $Q^*[\mathcal{H}_v]$ is connected[6], where $\mathcal{H}_v = \{H \in \mathcal{H} : H \ni v\}$, i.e., a support for the *dual hypergraph* $(\mathcal{H}, \{\mathcal{H}_v\}_{v \in V(G)})$. For a graph G and two families of connected subgraphs $\mathcal{H}$ and $\mathcal{K}$ of G, let the 3-tuple $(G, \mathcal{H}, \mathcal{K})$ denote an *intersection system*. An *intersection support* is a graph $\tilde{Q}$ that is a support for the *intersection hypergraph* $(\mathcal{H}, \{\mathcal{H}_K\}_{K \in \mathcal{K}})$, where $\mathcal{H}_K = \{H \in \mathcal{H} : K \cap H \neq \emptyset\}$. The notion of an intersection hypergraph generalizes both the primal and dual hypergraphs defined above.

3 Bounded genus graphs

Let $(G, \mathcal{H})$ be a graph system of genus g. Consider a cellular[7] embedding of G in an oriented surface of genus g. For a pair of subgraphs $H, H' \in \mathcal{H}$, we define the notion of a *reduced graph* that is required for the definition of a *cross-free* system.

Definition 2 (Reduced graph)**.** *Let $(G, \mathcal{H})$ be a graph system with G cellularly embedded in an oriented surface. For any two subgraphs $H, H' \in \mathcal{H}$, the* reduced graph $R(H, H')$ *is the embedded graph obtained from G by contracting all edges, both of whose end-points lie in $H \cap H'$, where multi-edges and self-loops are retained.*

Note that if G can be embedded in a surface Σ, then so can be $R(H, H')$.

Definition 3 (Cross-free at v)**.** *A graph system $(G, \mathcal{H})$ with G cellularly embedded in an oriented surface, is cross-free at a vertex $v \in V(G)$ if for any two subgraphs $H, H' \in \mathcal{H}_v$, the following holds: Let $\tilde{v}$ be the image of v in the reduced graph $R(H, H')$. Then, there are no 4 edges $e_i = \{\tilde{v}, v_i\}$ in $R(H, H')$, $i = 1, \ldots, 4$ incident to $\tilde{v}$ in cyclic order around $\tilde{v}$, s.t. $v_1, v_3 \in H \setminus H'$, and $v_2, v_4 \in H' \setminus H$.*

If there is an embedding of G s.t. $(G, \mathcal{H})$ is cross-free at every vertex of G, we say that $(G, \mathcal{H})$ is cross-free.

By the Jordan curve theorem, it follows that if $(G, \mathcal{H})$ is a non-piercing planar system, then the graph system $(G, \mathcal{H})$ is cross-free. It is easy to construct examples to show that the reverse direction does not hold in the plane.

[5] Note that we cannot simply project each H on $\mathbf{b}(V)$ as the resulting subgraphs may not be connected in G.

[6] To make the definition symmetric, we could have considered a coloring $c : \mathcal{H} \to \{\mathbf{r}, \mathbf{b}\}$, and required that $Q^*[\mathcal{H}^{\mathbf{b}_v}]$ be connected for each $v \in V$, where $\mathcal{H}^{\mathbf{b}_v} = \{H \in \mathcal{H} : H \ni v \text{ and } c(H) = \mathbf{b}\}$. However, this problem reduces to constructing a dual support restricted to the hypergraphs $\mathcal{H}^{\mathbf{b}} = \{H \in \mathcal{H} : c(H) = \mathbf{b}\}$. Therefore, in the dual setting, it is sufficient to study the uncolored version of the problem.

[7] A cellular, or 2-cell embedding of a graph G on a surface is an embedding where the edges are non-crossing, and each face is homeomorphic to a disk.

Theorem 4. *Let $(G, \mathcal{H})$ be a cross-free system of genus g, with $c : V \to \{\mathbf{r}, \mathbf{b}\}$. Then, there is a support Q of genus at most g on $\mathbf{b}(V)$ i.e., $Q[\mathbf{b}(H)]$ is connected for each $H \in \mathcal{H}$.*

Theorem 5. *Let $(G, \mathcal{H})$ be a cross-free system of genus g, then, there is a support Q^* on $\mathcal{H}$ of genus at most g i.e., $Q^*[\mathcal{H}_v]$ is connected for each $v \in V$.*

Theorem 6. *Let $(G, \mathcal{H}, \mathcal{K})$ be a cross-free intersection system of genus g. Then, there exists an intersection support $\tilde{Q}$ on $\mathcal{H}$ of genus at most g.*

In all the results above, we use the notion of *Vertex Bypassing* defined below:

Definition 7. *Let G be embedded in an oriented surface Σ. Let $N(v) = (v_1, \ldots, v_k, v_1)$ be the cyclic order of vertices around v. The Vertex Bypassing of v is defined as follows:*

1. *Subdivide each edge $\{v, v_i\}$ by a vertex u_i. Construct a cycle $C = (u_1, \ldots, u_k, u_1)$ by joining consecutive vertices u_i, u_{i+1} (with indices taken $\mod k$) by a simple arc not intersecting the edges of G s.t. the resulting graph remains embedded in Σ. Remove the vertex v. Let G'' denote the resulting graph.*

2. *$\forall\ H \in \mathcal{H}_v$, s.t. $\{v, v_i\} \in H$, let H'' denote the subgraph of G'' on $(H \setminus \{v\}) \cup \{\cup_{v_i \in H''} u_i\}$ Let $\mathcal{H}''_v = \{H'' : H \in \mathcal{H}_v\}$. Let $\mathcal{H}'' = (\mathcal{H} \setminus \mathcal{H}_v) \cup \mathcal{H}''_v$ (Note that the subgraphs in $\mathcal{H}'_v$ may not be connected).*

3. *Add a set D of non-intersecting chords[8] in C so that $\forall\ H \in \mathcal{H}''$, H induces a connected subgraph in $C \cup D$, and the resulting subgraphs remain cross-free.*

Let $(G', \mathcal{H}')$ be the resulting system.

The heart of the proof is in showing that Step 3 can be done, i.e., there exists a set of non-intersecting chords that we can add in C so that the resulting subgraphs are connected, and the system remains cross-free. Assuming we can apply vertex bypassing, the proof of Theorem 5 follows by repeatedly applying vertex bypassing to a vertex of *maximum depth* in G, i.e., to a vertex v in G maximizing $|\{H \in \mathcal{H} : H \ni v\}|$, until each vertex of the graph is in at most one subgraph. We can then obtain a support by contracting the edges in each subgraph. The proof of Theorem 6 follows by using Theorem 5 and techniques from the proof of Theorem 4.

4 Bounded Treewidth graphs

We show that if $(G, \mathcal{H})$ is a graph system, and $\mathcal{H}$ is a collection of non-piercing subgraphs then both the primal and dual supports have treewidth $O(2^{tw(G)})$ and this exponential blow-up in the treewidth of the support is sometimes necessary.

[8] We use the term non-intersecting to mean internally non-intersecting.

Theorem 8. *Let $(G, \mathcal{H})$ be a non-piercing graph system. Let $c : V(G) \to \{\mathbf{r}, \mathbf{b}\}$ be a 2-coloring of the vertices $V(G)$ of G. Then, there is a support Q on $\mathbf{b}(V)$ s.t. $tw(Q) \leq 3 \cdot 2^{tw(G)}$. Further, Q can be computed in time polynomial in $|G|, |H|$ if G has bounded treewidth. There exist non-piercing graph systems $(G, \mathcal{H})$ where any support has size $\Omega(2^{tw(G)})$.*

Theorem 9. *Let $(G, \mathcal{H})$ be a non-piercing graph system. There is a dual support Q^* on $\mathcal{H}$ s.t. $tw(Q^*) \leq 4 \cdot 2^{tw(G)}$. Further, Q^* can be computed in time polynomial in $|G|, |\mathcal{H}|$ if G has bounded treewidth. There exist non-piercing graph systems $(G, \mathcal{H})$ where any dual support has size $\Omega(2^{tw(G)})$.*

5 Outerplanar Graphs

Let $(G, \mathcal{H})$ be an outerplanar graph system. In the setting of outerplanar graphs, there is a difference between the primal and dual settings. In the primal setting, if the subgraphs are cross-free, then there is a primal support that is also outerplanar. In the dual setting, the cross-free condition is not sufficient. We show an example below. However, restricting the subgraphs to be non-piercing is sufficient for the system to admit a dual outerplanar support.

Consider a triangle drawn in the plane (with straight-line segments) with vertices $\{1, 2, 3\}$. Subdivide the segments $\{1, 2\}, \{2, 3\}$ and $\{1, 3\}$ by points $4, 5$, and 6 respectively. Add a triangle on the points $4, 5$ and 6. This defines an embedding of an asteroid-triple G. The subgraphs $\mathcal{H}$ are those by the points $\{1, 4, 2\}, \{2, 5, 3\}, \{1, 6, 3\}$ and $\{4, 5, 6\}$. It is easy to check that $(G, \mathcal{H})$ is cross free, and the support for the dual is K_4 which is not outerplanar.

Theorem 10. *Let $(G, \mathcal{H})$ be an outerplanar cross-free system, with $c : V(G) \to \{\mathbf{r}, \mathbf{b}\}$, a 2-coloring of the vertices $V(G)$ of G. Then, there is an outerplanar support Q on $\mathbf{b}(V)$ i.e., $Q[\mathbf{b}(H)]$ is connected for each $H \in \mathcal{H}$.*

Theorem 11. *Let $(G, \mathcal{H})$ be a non-piercing outerplanar system. Then, there is an outerplanar dual support Q^* on $\mathcal{H}$.*

6 Applications

In this section, we describe some applications of the existence of supports. Raman and Ray [27] showed that for an intersection hypergraph defined on a set of non-piercing regions in the plane, there is a planar support (See [27] for precise definitions), which implies a support for both the primal and dual settings for the hypergraphs defined by points and non-piercing regions in the plane.

Since graphs of genus g admit separators of size $O(\sqrt{gn})$ [15], all the algorithmic consequence of [27] generalize to cross-free systems on bounded genus graphs. Instead of describing a long sequence of results that follow from the existence of supports, we highlight just three results that follow as a consequence of Theorem 6.

Theorem 12. *Let $(G, \mathcal{H})$ be a cross-free system of genus g, then there exists*

1. *a PTAS for the Dominating Set problem, i.e., find $\mathcal{H}' \subseteq \mathcal{H}$ of minimum cardinality s.t. for each $H \in \mathcal{H}$, either $H \in \mathcal{H}'$ or $H \cap H' \neq \emptyset$ for some $H' \in \mathcal{H}'$.*

2. *a PTAS for the problem of packing points when each subgraph $H \in \mathcal{H}$ has capacity D_H bounded by a constant, i.e., find $V' \subseteq V$ of maximum cardinality s.t. $|H \cap V'| \leq D_H$ for each $H \in \mathcal{H}$.*

3. *a PTAS for the problem of packing subgraphs when each vertex $v \in V$ has capacity D_v bounded by a constant, i.e., find $\mathcal{H}' \subseteq \mathcal{H}$ of maximum cardinality s.t. $|\{H \in \mathcal{H}' : H \ni v\}| \leq D_v$ for each $v \in V$.*

Keller and Smorodinsky [19] showed that the intersection hypergraph of disks in the plane can be colored with 4 colors, and this was generalized by Keszegh [20] for pseudodisks, which was further generalized in [27] to show that the intersection hypergraph of non-piercing regions is 4-colorable. As a consequence of Theorem 6, we obtain the following.

Theorem 13. *Let $(G, \mathcal{H}, \mathcal{K})$ be a cross-free intersection system of genus g. Then, $\mathcal{H}$ can be colored with at most $\frac{7+\sqrt{1+24g}}{2}$ colors such that no hyperedge $\mathcal{H}_K$ is monochromatic.*

Proof. By Theorem 6, $(G, \mathcal{H}, \mathcal{K})$ has a support $\tilde{Q}$ of genus at most g. Now, $\chi(\tilde{Q}) \leq \frac{7+\sqrt{1+24g}}{2}$ [14]. Since $\tilde{Q}$ is a support, for each $K \in \mathcal{K}$, there is an edge between some two subgraphs $H, H' \in \mathcal{H}_K$. Therefore, no hyperedge $\mathcal{H}_K$ is monochromatic. □

Keszegh and Pàlvölgyi [21] introduced the notion of $ABAB$-free hypergraphs. Ackerman et al., [1] show that these are equivalent to hypergraphs with a *stabbed pseudo-disk representation*, i.e., each $S \in \mathcal{S}$ is mapped to a closed and bounded region D_S containing the origin whose boundary is a simple Jordan curve, each $x \in X$ is mapped to a point p_x in $\mathbb{R}^2$ s.t. $p_x \in D_S$ iff $x \in S$. The regions $\mathcal{D} = \{D_S : S \in \mathcal{S}\}$ form a stabbed pseudodisk arrangement. Let $(P, \mathcal{D})$ denote the embedding of the hypergraph where $P = \{p_x : x \in X\}$.

The authors show that to any stabbed pseudodisk arrangement $\mathcal{D}$ and a set P of points, we can add additional pseudodisks $\mathcal{D}'$ s.t. (i) each $D' \in \mathcal{D}'$ contains exactly 2 points of P, (ii) $\mathcal{D} \cup \mathcal{D}'$ is a pseudodisk arrangement, and (iii) Each $D \in \mathcal{D}$ s.t. $|D \cap P| \geq 3$ contains a pseudodisk $D' \in \mathcal{D}'$. The graph on P whose edges are defined by $\mathcal{D}'$ is called the *delaunay graph* of the arrangement. They show that $ABAB$-free hypergaphs are 3 colorable by showing that the delaunay graph is outerplanar. This result follows from Theorem 10 since a support for cross-free outerplanar graph system satisfies the properties of delaunay graph above.

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