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On asymptotic confirmation of the Faudree–Lehel Conjecture on the irregularity strength of graphs

(Extended abstract)

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Abstract

We call a multigraph irregular if it has pairwise distinct vertex degrees. No non-trivial (simple) graph is thus irregular. The irregularity strength of a graph G, $s(G)$, is a specific measure of the "level of irregularity" of G. It might be defined as the least k such that one may obtain an irregular multigraph of G by multiplying any selected edges of G, each into at most k its copies. In other words, $s(G)$ is the least k admitting a $\{1, 2, \ldots, k\}$-weighting of the edges of G assuring distinct weighted degrees for all the vertices, where the weighted degree of a vertex is the sum of its incident weights. The most well-known open problem concerning this graph invariant is the conjecture posed in 1987 by Faudree and Lehel that there exists an absolute constant C such that $s(G) \leq \frac{n}{d} + C$ for each d-regular graph G with n vertices and $d \geq 2$, whereas a straightforward counting argument implies that $s(G) \geq \frac{n}{d} + \frac{d-1}{d}$. Until very recently this conjecture had remained widely open. We shall discuss recent results confirming it asymptotically, up to a lower order term. If time permits we shall also mention a few related problems, such as the 1–2–3 Conjecture or the concept of irregular subgraphs, introduced recently by Alon and Wei, and progress in research concerning these.

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1 Introduction

One of the most basic facts in graph theory is that every simple graph of order at least two contains a pair of vertices with equal degrees. Thus it cannot be irregular, where by *irregular* we mean: containing pairwise distinct vertex degrees. There are irregular multigraphs, though. In fact any (simple) graph $G = (V, E)$ can be turned into an irregular multigraph throughout multiplying some of its edges, as long as G has no isolated edge and at most one isolated vertex. The least k such that it is feasible using at most k copies of every edge is called the irregularity strength of G and denoted $s(G)$; we set $s(G) = \infty$ if this is not possible at all. Note that equivalently, $s(G)$ may be defined as the least positive integer k for which there is an edge k-weighting, that is a function $\omega : E \to \{1, 2, \ldots, k\}$ such that each vertex $v \in V$ is attributed a distinct weighted degree $d_\omega(v) := \sum_{u \in N(v)} \omega(uv)$. This graph invariant was introduced in the 80s by Chartrand et al. [11] in relation to research on the concept of irregular graphs of Chartrand, Erdős and Oellermann [10]. In general it is known that $s(G) \leq n-1$ for all graphs with n vertices for which the parameter is finite except K_3 [3, 31], and this upper bound is tight, e.g. for the family of stars. It can however be significantly decreased for graphs without small degree vertices. In particular, it is easy to verify that $s(G) \geq \frac{n}{d} + \frac{d-1}{d}$ for d-regular graphs, while the central open problem of this field is the following conjecture of Faudree and Lehel [17] from 1987 (posed first as a question by Jacobson, see [29]).

Conjecture 1. *There exists a constant C such that $s(G) \leq \frac{n}{d} + C$ for every d-regular graph G with $d \geq 2$ and order n.*

This problem "energized the study of the irregularity strength", as Cuckler and Lazebnik stated in [12], and still remains open. A significant step forward towards solving it was achieved in 2002 by Frieze, Gould, Karoński, and Pfender [20], who used the probabilistic method to prove the first linear bound $s(G) \leq 48(n/d) + 1$ for $d \leq \sqrt{n}$, and a super-linear one $s(G) \leq 240(\log n)(n/d) + 1$ in the remaining cases. They also proved similar bounds for general graphs, with d replaced by the minimum degree δ. For example, they showed that $s(G) = O(n/\delta)$ for the maximum degree $\Delta \leq n^{1/2}$. The linear bounds in n/δ was further extended to the case when $d \geq 10^{4/3} n^{2/3} \log^{1/3} n$ and $\delta \geq 10 n^{3/4} \log^{1/4} n$, respectively, by Cuckler and Lazebnik [12]. The first general and unified linear bound in n/δ for the full spectrum of (n, δ) was delivered by Przybyło [34, 35], who used a constructive approach to prove that $s(G) \leq 16(n/d) + 6$ and $s(G) \leq 112(n/\delta) + 28$, respectively. Since then several attempts based on inventive new algorithms have been conducted in pursuit towards improvement of the multiplicative constant in front of n/δ, see e.g. [23, 24, 30]. The best result among these is due to Kalkowski, Karoński, and Pfender [24], who invented a deterministic algorithm implying that in general, $s(G) \leq 6\lceil n/\delta \rceil$ for graphs with minimum degree $\delta \geq 1$ and without isolated edges. Conjecture 1 throughout more than 35 years since its formulation was an inspiration for many results, see e.g. [3, 5, 8, 12, 13, 15, 16, 17, 18, 20, 22, 24, 30, 31, 33, 34, 35], and various related problems and concepts, giving rise to a reach and vital branch of graph theory, see [21, 29] for surveys devoted to some of them. Only just recently it was proved by Przybyło [32] that the Faudree-Lehel Conjecture holds asymptotically almost surely for random graphs $G(n,p)$ (which are typically "close to" regular graphs), for any constant p, and holds asymptotically for a wide spectrum of values of d [33].

2 Main Results

Developing research from [33], we managed to confirm asymptotically, up to a lower order term, an extension of Conjecture 1 towards the setting of general graphs.

Theorem 2 ([38])**.** *For every $\varepsilon \in (0, 0.25)$, there are absolute constants C_1, C_2 such that for each graph G with n vertices and minimum degree $\delta > 0$ which does not contain isolated edges, $s(G) \leq \frac{n}{\delta}(1 + \frac{C_1}{\delta^\varepsilon}) + C_2$.*

We also confirmed that the generalization of Faudree-Lehel Conjecture holds, not only asymptotically, for relatively dense graphs.

Theorem 3 ([38])**.** *For every $0.8 < \alpha \leq 1$, there is an absolute constant C such that for each graph G with n vertices and minimum degree $\delta \geq n^\alpha$, $s(G) \leq \frac{n}{\delta} + C$.*

In the case of regular graphs exclusively, we also provided a much shorter argument, implying a more specific result directly related with Conjecture 1.

Theorem 4 ([39])**.** *Given any $\varepsilon \in (0, 0.25)$, for every d-regular graph G with n vertices, if d is sufficiently large in terms of ε, $s(G) < \frac{n}{d}(1 + \frac{14}{d^\varepsilon}) + 28$.*

Theorem 5 ([39])**.** *Given any $0.8 < \alpha \leq 1$, for every d-regular graph G on n vertices with $d \geq n^\alpha$, if d is sufficiently large in terms of α, then $s(G) < \frac{n}{d} + 28$.*

3 Main Ideas

3.1 General Graphs

A very vague general idea behind our construction yielding Theorems 2 and 3 is to randomly partition V into a big set B and a small set S, where $|S| = (n/\delta) \cdot o(\delta)$, in a special and controlled manner. We then first randomly modify the edge weights so that almost all vertices in B have distinct weighted degrees. Finally, we locally adjust weighted degrees of the rest of the vertices in order to differentiate them in entire G.

Our approach can be divided into three main steps.

Step 1 relies on a random construction assuring relatively sparse distribution of weighted degrees of the vertices in B, i.e. without too many vertex weights in any of the predefined intervals partitioning positive integers. A general, yet still imprecise idea here is to assign to every vertex v a random variable $X_v \sim U[0, 1]$, and then attribute an edge uv a small weight if $X_u + X_v$ is small, and a large weight, otherwise. This way a small value of X_v

pulls the weighted degree of v downwards, while a large value of X_v pushes its weighted degree up.

Step 2 concentrates around modifications of weights of the edges between B and S, resulting in relatively small weights' shifts, attributing pairwise distinct weighted degrees to all but a small set of "bad vertices" in B. Note that in order to be able to achieve our goal, we must assure that the (randomly chosen) set S is large enough to guarantee sufficiently many edges between B and S.

In **Step 3** we modify mainly weights of the edges within S (and a small fraction of the edges outside S) in order to differentiate weighted degrees in S mostly. For this purpose we associate to these vertices special weighted degrees, which were earlier deliberately not used within step 2. While distinguishing weighted degrees in S we in particular benefit from the fact that S is small in comparison to B, and thus vertices in S have on average large fraction of their incident edges in $E(S, B)$ (statistically much larger than the fraction of edges in S). This allows taking on essential preparatory measures prior to step 3 (in step 1) assuring sparse weighted degrees' distribution within S and facilitating the mentioned final cleanup in this set. Throughout the construction we moreover specify several types of "bad vertices", which do not fulfill one of a list of certain conditions and cannot be distinguished according to major procedures. The aggregated set of these is however small enough to be taken care of in a special manner within step 3.

3.2 Regular Graphs

In order to provide much shorter proof of more specific results in the case of regular graphs, i.e. Theorems 4 and 5 (directly referring to Conjecture 1), we use in a way similar general 3-step approach, exploiting in particular random variables $X_v \sim U[0, 1]$ associated with vertices. We however phrase our construction differently, using quantization and the Lovász Local Lemma, which was redundant in the construction above. This time we may guarantee that weighted degrees of the vertices in the big set B are arranged very tightly, in fact these form a sequence of $|B|$ consecutive integers. We moreover again benefit from S being small compared to B, this time by assigning heavy weights between S and B, thus guaranteeing that weighted degrees of vertices in (the small set) S are all larger than those in B (as random choice of S and B results, with positive probability, in many edges joining vertices in S with those in B). Still particular preparatory measures need to be undertaken within our special initial random vertex and edge partitions, in order to facilitate later final weighted degrees distinction within S. We refer the reader to [38, 39] for more details of our randomized constructions.

4 Related Concepts

One of the most well known variants of the irregularity strength is its local correspondent, within which one confines to requiring distinct weighted degrees only for adjacent vertices. This concept was introduced in 2004 by Karoński, Łuczak and Thomason [26] together with an intriguing conjecture that just weights 1, 2 and 3 are sufficient for every graphs without isolated edges within such a setting. This so-called 1–2–3 Conjecture swiftly became yet another central problem of this field, and gained considerable attention, comparable to the Conjecture of Faudree and Lehel, cf. in particular [1, 2, 6, 7, 9, 14, 25, 26, 27, 28, 36, 37, 40, 41, 42, 43, 44]. In 2021 the 1–2–3 Conjecture was proven to hold for regular graphs with large enough degrees [36], while in 2022 also for general graphs with minimum degree $\delta = \Omega(\log \Delta)$ [37]. Lately Keusch [28] proved that actually weights $1, 2, 3, 4$ always suffice, whereas very recently the same author announced [27] to finally resolve the conjecture in the affirmative.

Also recently yet another related concept was proposed by Alon and Wei [4]. Roughly speaking they posed a conjecture that every graph contains a spanning subgraph which is (globally) almost as irregular as possible. More precisely they asked if any d-regular graph on n vertices contains a spanning subgraph in which the number of vertices of each degree between 0 and d deviates from $\frac{n}{d+1}$ by at most 2, and similarly, if every graph on n vertices, not necessarily regular, with minimum degree δ contains a spanning subgraph in which the number of vertices of each degree does not exceed $\frac{n}{\delta+1} + 2$. They also supported the conjectures by showing in particular that if $d^3 \log n \leq o(n)$ then every d-regular graph with n vertices contains a spanning subgraph in which the number of vertices of each degree between 0 and d is $(1 + o(1))\frac{n}{d+1}$, and a similar result for general graphs. Some of these results were also later significantly strengthened by Fox, Luo and Pham [19].

The mentioned problems are just the tip of the iceberg of related concepts. An extensive list of other related issues can in particular be found in Gallian's survey [21].

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