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# Strengthening the Directed Brooks' Theorem for oriented graphs and consequences on digraph redicolouring

(Extended abstract)

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**Abstract**

Let $D = (V, A)$ be a digraph. We define $\Delta_{\max}(D)$ as the maximum of $\{\max(d^+(v), d^-(v)) \mid v \in V\}$ and $\Delta_{\min}(D)$ as the maximum of $\{\min(d^+(v), d^-(v)) \mid v \in V\}$. It is known that the dichromatic number of $D$ is at most $\Delta_{\min}(D) + 1$. In this work, we prove that every digraph $D$ which has dichromatic number exactly $\Delta_{\min}(D) + 1$ must contain the directed join of $\overleftrightarrow{K_r}$ and $\overleftrightarrow{K_s}$ for some $r, s$ such that $r + s = \Delta_{\min}(D) + 1$, except if $\Delta_{\min}(D) = 2$ in which case $D$ must contain a digon. In particular, every oriented graph $\vec{G}$ with $\Delta_{\min}(\vec{G}) \geq 2$ has dichromatic number at most $\Delta_{\min}(\vec{G})$.

Let $\vec{G}$ be an oriented graph of order $n$ such that $\Delta_{\min}(\vec{G}) \leq 1$. Given two 2-dicolourings of $\vec{G}$, we show that we can transform one into the other in at most $n$ steps, by recolouring one vertex at each step while maintaining a dicolouring at any step. Furthermore, we prove that, for every oriented graph $\vec{G}$ on $n$ vertices, the distance between two $k$-dicolourings is at most $2\Delta_{\min}(\vec{G})n$ when $k \geq \Delta_{\min}(\vec{G}) + 1$.

We then extend a theorem of Feghali, Johnson and Paulusma to digraphs. We prove that, for every digraph $D$ with $\Delta_{\max}(D) = \Delta \geq 3$ and every $k \geq \Delta + 1$, the $k$-dicolouring graph of $D$ consists of isolated vertices and at most one further component that has diameter at most $c_\Delta n^2$, where $c_\Delta = O(\Delta^2)$ is a constant depending only on $\Delta$.

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# 1 Introduction

## 1.1 Graph (re)colouring

Given a graph $G = (V, E)$, a *k-colouring* of $G$ is a function $c : V \to \{1, \ldots, k\}$ such that, for every edge $xy \in E$, we have $c(x) \neq c(y)$. The *chromatic number* of $G$, denoted by $\chi(G)$, is the smallest $k$ such that $G$ admits a $k$-colouring. The *maximum degree* of $G$, denoted by $\Delta(G)$, is the degree of the vertex with the greatest number of edges incident to it. A simple greedy procedure shows that, for any graph $G$, $\chi(G) \leq \Delta(G) + 1$. The celebrated theorem of Brooks [6] characterizes the graphs for which equality holds.

**Theorem 1** ([6])**.** *A connected graph $G$ satisfies $\chi(G) = \Delta(G) + 1$ if and only if $G$ is an odd cycle or a complete graph.*

For any $k \geq \chi(G)$, the *k-colouring graph* of $G$, denoted by $\mathcal{C}_k(G)$, is the graph whose vertices are the $k$-colourings of $G$ and in which two $k$-colourings are adjacent if they differ by the colour of exactly one vertex. A path between two given colourings in $\mathcal{C}_k(G)$ corresponds to a *recolouring sequence*. In the last fifteen years, since the papers of Bonsma, Cereceda, van den Heuvel and Johnson [7, 3, 8, 9], graph recolouring has been studied by many researchers in graph theory. Feghali, Johnson and Paulusma [10] proved the following analogue of Brooks' Theorem for graphs recolouring.

**Theorem 2** ([10])**.** *Let $G = (V, E)$ be a connected graph with $\Delta(G) = \Delta \geq 3$, $k \geq \Delta + 1$, and $\alpha$, $\beta$ two $k$-colourings of $G$. Then at least one of the following holds:*

- *$\alpha$ or $\beta$ is an isolated vertex in $\mathcal{C}_k(G)$, or*
- *there is a recolouring sequence of length at most $c_\Delta |V|^2$ between $\alpha$ and $\beta$, where $c_\Delta = O(\Delta)$ is a constant depending on $\Delta$.*

Considering graphs of bounded maximum degree, Theorem 2 has been very recently improved by Bousquet, Feuilloley, Heinrich and Rabie (see [4]). They prove that there is a recolouring sequence between $\alpha$ and $\beta$ of length at most $f(\Delta)|V|$ for some computable function $f$, except if $\alpha$ or $\beta$ is an isolated vertex in $\mathcal{C}_k(G)$.

## 1.2 Digraph (re)dicolouring

In this paper, we are looking for extensions of the previous results on graphs colouring and recolouring to digraphs.

Let $D$ be a digraph. A *digon* is a pair of arcs in opposite directions between the same vertices. An *oriented graph* is a digraph with no digon. The *bidirected graph* associated to a graph $G$, denoted by $\overleftrightarrow{G}$, is the digraph obtained from $G$, by replacing every edge by a digon. The *underlying graph* of $D$, denoted by $UG(D)$, is the undirected graph $G$ with vertex set $V(D)$ in which $uv$ is an edge if and only if $uv$ or $vu$ is an arc of $D$.

Let $v$ be a vertex of a digraph $D$. The *out-degree* (resp. *in-degree*) of $v$, denoted by $d^+(v)$ (resp. $d^-(v)$), is the number of arcs leaving (resp. entering) $v$. We define the *maximum degree* of $v$ as $d_{\max}(v) = \max\{d^+(v), d^-(v)\}$, and the *minimum degree* of $v$ as $d_{\min}(v) = \min\{d^+(v), d^-(v)\}$. We can then define the corresponding maximum degrees of $D$: $\Delta_{\max}(D) = \max_{v \in V(D)}(d_{\max}(v))$ and $\Delta_{\min}(D) = \max_{v \in V(D)}(d_{\min}(v))$. A digraph $D$ is $\Delta$-*diregular* if, for every vertex $v \in V(D)$, $d^-(v) = d^+(v) = \Delta$. The *directed join* of $D_1$ and $D_2$, denoted by $D_1 \Rightarrow D_2$, is the digraph obtained from disjoint copies of $D_1$ and $D_2$ by adding all arcs from the copy of $D_1$ to the copy of $D_2$.

In 1982, Neumann-Lara [12] introduced the notions of dicolouring and dichromatic number, which generalize the ones of colouring and chromatic number. A *k-dicolouring* of $D$ is a function $c : V(D) \to \{1, \dots, k\}$ such that $c^{-1}(i)$ induces an acyclic subdigraph in $D$ for each $i \in \{1, \dots, k\}$. The *dichromatic number* of $D$, denoted by $\vec{\chi}(D)$, is the smallest $k$ such that $D$ admits a $k$-dicolouring. There is a one-to-one correspondence between the $k$-colourings of a graph $G$ and the $k$-dicolourings of the associated bidirected graph $\overleftrightarrow{G}$, and in particular $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$. Hence every result on graph colourings can be seen as a result on dicolourings of bidirected graphs, and it is natural to study whether the result can be extended to all digraphs.

The directed version of Brooks' Theorem was first proved by Harutyunyan and Mohar in [11] (see also [1]).

**Theorem 3** (Directed Brooks' Theorem)**.** *Let $D$ be a connected digraph. Then $\vec{\chi}(D) \leq \Delta_{\max}(D) + 1$ and equality holds if and only if $D$ is a directed cycle, a bidirected odd cycle or a bidirected complete graph of order at least 4.*

It is easy to prove, by a simple greedy procedure, that every digraph $D$ can be dicoloured with $\Delta_{\min}(D) + 1$ colours. Hence, one can wonder if Brooks' Theorem can be extended to digraphs using $\Delta_{\min}(D)$ instead of $\Delta_{\max}(D)$. Our main result is the following.

**Theorem 4.** *Let $D$ be a digraph. If $\vec{\chi}(D) = \Delta_{\min}(D) + 1$, then one of the following holds:*

- $\Delta_{\min}(D) \leq 1$, *or*
- $\Delta_{\min}(D) = 2$ *and $D$ contains* $\overleftrightarrow{K_2}$, *or*
- $\Delta_{\min}(D) \geq 3$ *and $D$ contains* $\overleftrightarrow{K_r} \Rightarrow \overleftrightarrow{K_s}$, *for some $r, s \geq 0$ such that* $r + s = \Delta_{\min}(D) + 1$.

In particular, the following is a direct consequence of Theorem 4.

**Corollary 5.** *Let $D$ be a digraph. If $\vec{\chi}(D) = \Delta_{\min}(D) + 1$, then $D$ contains the complete bidirected graph on $\left\lceil \frac{\Delta_{\min}(D)+1}{2} \right\rceil$ vertices as a subdigraph.*

Corollary 5 is best possible: if we restrict $D$ to not contain the complete bidirected graph on $\left\lceil \frac{\Delta_{\min}(D)+1}{2} \right\rceil + 1$ vertices, then deciding $\vec{\chi}(D) \leq \Delta_{\min}(D)$ is NP-complete (see [13]). Since an oriented graph does not contain any digon, Corollary 5 directly implies the following.

**Corollary 6.** *Let $\vec{G}$ be an oriented graph. If $\Delta_{\min}(\vec{G}) \geq 2$, then $\vec{\chi}(\vec{G}) \leq \Delta_{\min}(\vec{G})$.*

For any $k \geq \vec{\chi}(D)$, the *k-dicolouring graph* of $D$, denoted by $\mathcal{D}_k(D)$, is the graph whose vertices are the $k$-dicolourings of $D$ and in which two $k$-dicolourings are adjacent if they differ by the colour of exactly one vertex. Observe that $\mathcal{C}_k(G) = \mathcal{D}_k(\overleftrightarrow{G})$ for any bidirected graph $\overleftrightarrow{G}$. A *redicolouring sequence* between two dicolourings is a path between these dicolourings in $\mathcal{D}_k(D)$.

Digraph redicolouring was first introduced in [5], where the authors generalized different results on graph recolouring to digraphs, and proved some specific results when restricted to oriented graphs. In particular, they studied the $k$-dicolouring graph of digraphs with bounded degeneracy or bounded maximum average degree, and they show that finding a redicolouring sequence between two given $k$-dicolourings of a digraph is PSPACE-complete for every fixed $k \geq 2$. Dealing with the maximum degree of a digraph, they proved that, given an orientation of a subcubic graph $\vec{G}$ on $n$ vertices, its 2-dicolouring graph $\mathcal{D}_2(\vec{G})$ is connected and has diameter at most $2n$ and they asked if this bound can be improved. We answer this question by proving the following theorem.

**Theorem 7.** *Let $\vec{G}$ be an oriented graph of order $n$ such that $\Delta_{\min}(\vec{G}) \leq 1$. Then $\mathcal{D}_2(\vec{G})$ is connected and has diameter exactly $n$.*

In particular, if $\vec{G}$ is an orientation of a subcubic graph, then $\Delta_{\min}(\vec{G}) \leq 1$ (because $d^+(v) + d^-(v) \leq 3$ for every vertex $v$), and so $\mathcal{D}_2(\vec{G})$ has diameter exactly $n$. Furthermore, we prove the following as a consequence of Corollary 6 and Theorem 7.

**Corollary 8.** *Let $\vec{G}$ be an oriented graph of order $n$ with $\Delta_{\min}(\vec{G}) = \Delta \geq 1$, and let $k \geq \Delta + 1$. Then $\mathcal{D}_k(\vec{G})$ is connected and has diameter at most $2\Delta n$.*

Corollary 8 does not hold for digraphs in general: indeed, $\overleftrightarrow{P_n}$, the bidirected path on $n$ vertices, satisfies $\Delta_{\min}(\overleftrightarrow{P_n}) = 2$ and $\mathcal{D}_3(\overleftrightarrow{P_n}) = \mathcal{C}_3(P_n)$ has diameter $\Omega(n^2)$, as proved in [2]. Our last result is the following extension of Theorem 2 to digraphs.

**Theorem 9.** *Let $D = (V, A)$ be a connected digraph with $\Delta_{\max}(D) = \Delta \geq 3$, $k \geq \Delta + 1$, and $\alpha$, $\beta$ two $k$-dicolourings of $D$. Then at least one of the following holds:*

- *$\alpha$ or $\beta$ is an isolated vertex in $\mathcal{D}_k(G)$, or*
- *there is a redicolouring sequence of length at most $c_\Delta |V|^2$ between $\alpha$ and $\beta$, where $c_\Delta = O(\Delta^2)$ is a constant depending only on $\Delta$.*

Furthermore we prove that $\mathcal{D}_k(D)$ has an isolated vertex if and only if $D$ is bidirected and its underlying graph has one. Thus, an obstruction in Theorem 9 is exactly the bidirected graph of an obstruction in Theorem 2.

In the next section we prove Theorem 4. The integrality of the proofs of the results in this extended abstract can be found in [13].

## 2 Proof of Theorem 4

A digraph $D$ is *$k$-dicritical* if $\vec{\chi}(D) = k$ and for every vertex $v \in V(D)$, $\vec{\chi}(D - v) < k$. Observe that every digraph with dichromatic number at least $k$ contains a $k$-dicritical subdigraph. Let $\mathcal{F}_2$ be $\{\overleftrightarrow{K_2}\}$, and for each $\Delta \geq 3$, we define $\mathcal{F}_\Delta = \{\overleftrightarrow{K_r} \Rightarrow \overleftrightarrow{K_s} \mid r, s \geq 0 \text{ and } r + s = \Delta + 1\}$. A digraph $D$ is *$\mathcal{F}_\Delta$-free* if it does not contain $F$ as a subdigraph, for any $F \in \mathcal{F}_\Delta$. Theorem 4 can then be reformulated as follows.

**Theorem 4.** *Let $D$ be a digraph with $\Delta_{\min}(D) = \Delta \geq 2$. If $D$ is $\mathcal{F}_\Delta$-free, then $\vec{\chi}(D) \leq \Delta$.*

*Proof.* Let $D$ be a digraph such that $\Delta_{\min}(D) = \Delta \geq 2$ and $\vec{\chi}(D) = \Delta + 1$. We will show that $D$ contains some $F \in \mathcal{F}_\Delta$ as a subdigraph.

Let $(X, Y)$ be a partition of $V(D)$ such that for each $x \in X$, $d^+(x) \leq \Delta$, and for each $y \in Y$, $d^-(y) \leq \Delta$. We define the digraph $\tilde{D}$ as follows:

- $V(\tilde{D}) = V(D)$,
- $A(\tilde{D}) = A(D\langle X\rangle) \cup A(D\langle Y\rangle) \cup \{xy, yx \mid xy \in A(D), x \in X, y \in Y\}$.

Let us first prove that $\vec{\chi}(\tilde{D}) \geq \Delta + 1$. Assume for a contradiction that there exists a $\Delta$-dicolouring $c$ of $\tilde{D}$. Then $D$, coloured with $c$, must contain a monochromatic directed cycle $C$. Now $C$ is not contained in $X$ nor $Y$, for otherwise $C$ would be a monochromatic directed cycle of $D\langle X\rangle$ or $D\langle Y\rangle$ and so a monochromatic directed cycle of $\tilde{D}$. Thus $C$ contains an arc $xy$ from $X$ to $Y$. But then, $\{xy, yx\}$ is a monochromatic digon in $\tilde{D}$, a contradiction.

Since $\vec{\chi}(\tilde{D}) \geq \Delta + 1$, there is a $(\Delta + 1)$-dicritical subdigraph $H$ of $\tilde{D}$. By dicriticality of $H$, for every vertex $v \in V(H)$, $d_H^+(v) \geq \Delta$ and $d_H^-(v) \geq \Delta$, for otherwise a $\Delta$-dicolouring of $H - v$ could be extended to $H$ by choosing for $v$ a colour which is not appearing in its out-neighbourhood or in its in-neighbourhood. We define $X_H$ as $X \cap V(H)$ and $Y_H$ as $Y \cap V(H)$. Note that both $H\langle X_H\rangle$ and $H\langle Y_H\rangle$ are subdigraphs of $D$.

We will now prove that $H$ is $\Delta$-diregular. Let $\ell$ be the number of digons between $X_H$ and $Y_H$ in $H$. Observe that, by definition of $X$ and $H$, for each vertex $x \in X_H$, $d_H^+(x) = \Delta$. Note also that, in $H$, $\ell$ is exactly the number of arcs leaving $X_H$ and exactly the number of arcs entering $X_H$. We get:

$$\Delta|X_H| = \sum_{x \in X_H} d_H^+(x) = \ell + |A(H\langle X_H\rangle)| = \sum_{x \in X_H} d_H^-(x)$$

which implies, since $H$ is dicritical, $d_H^+(x) = d_H^-(x) = \Delta$ for every vertex $x \in X_H$. Using a symmetric argument, we prove that $\Delta|Y_H| = \sum_{y \in Y_H} d_H^+(y)$, implying $d_H^+(y) = d_H^-(y) = \Delta$ for every vertex $y \in Y_H$.

Since $H$ is $\Delta$-diregular, then in particular $\Delta_{\max}(H) = \Delta$. Hence, because $\vec{\chi}(H) = \Delta + 1$, by Theorem 3, either $\Delta = 2$ and $H$ is a bidirected odd cycle, or $\Delta \geq 3$ and $H$ is the bidirected complete graph on $\Delta + 1$ vertices. In both cases, $D\langle V(H)\rangle$ contains a digraph of $\mathcal{F}_\Delta$ as a subdigraph. □

# References

[1] Pierre Aboulker and Guillaume Aubian. Four proofs of the directed Brooks' Theorem. *arXiv preprint arXiv:2109.01600*, 2021.

[2] Marthe Bonamy, Matthew Johnson, Ioannis Lignos, Viresh Patel, and Daniël Paulusma. Reconfiguration graphs for vertex colourings of chordal and chordal bipartite graphs. *Journal of Combinatorial Optimization*, 27(1):132–143, 2014.

[3] Paul Bonsma and Luis Cereceda. Finding paths between graph colourings: PSPACE-completeness and superpolynomial distances. *Theoretical Computer Science*, 410(50):5215–5226, 2009. Mathematical Foundations of Computer Science (MFCS 2007).

[4] Nicolas Bousquet, Laurent Feuilloley, Marc Heinrich, and Mikaël Rabie. Short and local transformations between $(\Delta + 1)$-colorings. *arXiv preprint arXiv:2203.08885*, 2022.

[5] Nicolas Bousquet, Frédéric Havet, Nicolas Nisse, Lucas Picasarri-Arrieta, and Amadeus Reinald. Digraph redicolouring. *arXiv preprint arXiv:2301.03417*, 2023.

[6] Rowland Leonard Brooks. On colouring the nodes of a network. *Mathematical Proceedings of the Cambridge Philosophical Society*, 37(2):194–197, 1941.

[7] Luis Cereceda, Jan van den Heuvel, and Matthew Johnson. Connectedness of the graph of vertex-colourings. *Discrete Mathematics*, 308(5):913–919, 2008. Selected Papers from 20th British Combinatorial Conference.

[8] Luis Cereceda, Jan Van den Heuvel, and Matthew Johnson. Mixing 3-colourings in bipartite graphs. *European Journal of Combinatorics*, 30(7):1593–1606, 2009.

[9] Luis Cereceda, Jan van den Heuvel, and Matthew Johnson. Finding paths between 3-colorings. *Journal of Graph Theory*, 67(1):69–82, 2011.

[10] Carl Feghali, Matthew Johnson, and Daniël Paulusma. A reconfigurations analogue of Brooks' Theorem and its consequences. *Journal of Graph Theory*, 83(4):340–358, 2016.

[11] Ararat Harutyunyan and Bojan Mohar. Gallai's theorem for list coloring of digraphs. *SIAM Journal on Discrete Mathematics*, 25(1):170–180, 2011.

[12] Victor Neumann-Lara. The dichromatic number of a digraph. *J. Combin. Theory Ser. B.*, 33:265–270, 1982.

[13] Lucas Picasarri-Arrieta. Strengthening the Directed Brooks' Theorem for oriented graphs and consequences on digraph redicolouring. *arXiv preprint arXiv:2301.04881*, 2023.