

MINIMUM VERTEX DEGREE CONDITIONS FOR LOOSE SPANNING TREES IN 3-GRAPHS

(EXTENDED ABSTRACT)

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Abstract

In 1995, Komlós, Sárközy and Szemerédi showed that for large n , every n -vertex graph with minimum degree at least $(1/2 + \gamma)n$ contains all spanning trees of bounded degree. We consider a generalization of this result to loose spanning hypertrees, that is, linear hypergraphs obtained by successively appending edges sharing a single vertex with a previous edge, in 3-graphs. We show that for all γ and Δ , and n large, every n -vertex 3-uniform hypergraph of minimum vertex degree $(5/9 + \gamma)\binom{n}{2}$ contains every loose spanning tree with maximum vertex degree Δ . This bound is asymptotically tight, since some loose trees contain perfect matchings.

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1 Introduction

A classical result of Komlós, Sárközy and Szemerédi [4] states that for large n , any n -vertex graph with minimum degree $(1/2 + \varepsilon)n$ contains every spanning tree of bounded degree. Since a Hamilton path is a tree of bounded degree, the constant $1/2$ is best possible by any construction showing that Dirac's theorem is best possible.

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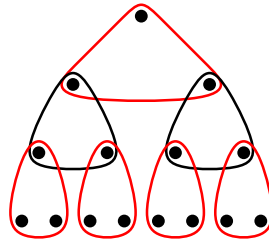


Figure 1: The complete binary loose tree with 4 levels. A perfect matching is shown in red.

We consider the corresponding problem for hypergraphs. We work with a combinatorial definition of a k -uniform ℓ -tree – a k -uniform hypergraph admitting an edge ordering e_1, \dots, e_m such that each e_i shares ℓ vertices with one previous edge in the ordering¹. Such orderings we call *valid*, and the edges which can be last in a valid ordering we call *leaves*. We will call 1-trees *loose* (also known in the literature as *linear*). Similarly, a $(k-1)$ -tree is also called a *tight tree*. We denote by $\delta_\ell(H)$ the *minimum ℓ -degree* of a k -graph H , that is, the minimum number of edges containing a set of ℓ vertices of H . Maximum degree is defined accordingly.

Not much is known about extensions of Komlós, Sárközy and Szemerédi's result to general k -uniform ℓ -trees, apart from a recent result of Pavez-Signé, Sanhueza-Matamala and Stein [5, 6] which shows that minimum $\delta_{k-1}(H) \geq (1/2 + \gamma)n$ forces the existence of any tight spanning tree T with $\Delta_1(T) \leq \Delta$.

Buß, Hàn and Schacht [1] showed that if $\delta_1(H) \geq (\frac{7}{16} + \varepsilon) \binom{n}{2}$, then H contains a loose Hamilton cycle – a cycle whose adjacent edges share exactly one vertex. The constant $7/16$ is best possible, and in a later paper Han and Zhao [3] gave the exact threshold.

In light of this, one may conjecture that 3-graphs with minimum vertex degree $(\frac{7}{16} + \varepsilon) \binom{n}{2}$ also contain every loose tree of bounded degree. However, this is not the case. Consider the *complete binary loose tree* as shown in Figure 1. A complete binary loose tree T_b with an even number of levels contains a perfect matching, so any 3-graph without a perfect matching will also not contain T_b . The asymptotic minimum degree threshold for perfect matchings in 3-graphs was shown to be $5/9$ by Hàn, Person and Schacht [2]. Their asymptotic bound was later made exact by Treglown, Kühn and Osthus [8]. This is tight as witnessed by the hypergraph on vertex set $A \cup B$ with $|A| = n/3 - 1$ and $|B| = 2n/3 + 1$ consisting of all edges with at least one vertex in A . Therefore, the minimum vertex degree threshold for the existence of bounded degree loose spanning trees must be at least $5/9$. We show that this is in fact the correct threshold.

Theorem 1.1. *For all $\gamma > 0$ and $\Delta \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that any 3-graph H on $n \geq n_0$ vertices with n odd and $\delta_1(H) \geq (\frac{5}{9} + \gamma) \binom{n}{2}$ contains every n -vertex loose tree T with $\Delta_1(T) \leq \Delta$.*

¹More formally, for each $i \geq 2$ there exists $j < i$ such that $e_i \cap \bigcup_{j' < i} e_{j'} \subseteq e_j$ and $|e_i \cap e_j| = \ell$.

2 Proof of Theorem 1.1

Our proof employs a classic recipe prescribed by the absorbing method.

Step 1. Find an *absorbing set* \mathcal{A} in our host graph. In Absorbing set lemma we show existence and in Absorbing lemma we show its absorbing properties.

Step 2. Embed a small proportion of our tree T in a way that covers the relevant vertices of \mathcal{A} . This is Covering lemma.

Step 3. Use the regularity method to extend this embedding to almost all of T . This is Approximate embedding lemma.

Step 4. Use \mathcal{A} to extend the embedding to all of T .

As an expansion of this sketch we give statements and proof ideas of the four lemmas used above, and show how they imply Theorem 1.1.

The proof of the following lemma is analogous to [6, Proposition 9.4 and Lemma 9.5]. It uses the fact that in a graph with minimum vertex degree $(\frac{1}{2} + o(1)) \binom{n}{2}$ every triple of vertices (w_1, w_2, w_3) has a positive density of absorbing pairs of Δ -stars (see Figure 2). We denote the set of such star-pairs by $A_\Delta(w_1, w_2, w_3)$. Subsampling these over all triples of vertices with the appropriate probability gives a large absorbing set.

Absorbing set lemma. *Let $1/n \ll \alpha \ll \beta \ll \gamma, 1/\Delta$. Let H be a 3-graph on n vertices with $\delta_1(H) \geq (\frac{1}{2} + \gamma) \binom{n}{2}$. Then there exists a set \mathcal{A} of at most βn vertex-disjoint pairs of Δ -stars such that for every triple (w_1, w_2, w_3) of distinct vertices in H we have $|A_\Delta(w_1, w_2, w_3) \cap \mathcal{A}| \geq \alpha n$.*

The following lemma shows that the set \mathcal{A} in fact absorbs – given a partial embedding of a tree which covers \mathcal{A} , we can use \mathcal{A} to find a full embedding. Intuitively, this is possible because given a triple (w_1, w_2, w_3) and one of its absorbing star-pairs (S_{u_2}, S_{u_3}) , we can add the edge $\{w_1, u_2, u_3\}$ to the partial embedding by switching u_2 for w_2 and u_3 for w_3 (see Figure 2). Repeating this switch enough times gives a full embedding of T .

Absorbing lemma. *Let $1/n \ll \eta < \alpha < 1/\Delta$. Let T be a loose 3-tree on n vertices of maximum degree Δ with a valid ordering of the edges $e_1, \dots, e_{(n-1)/2}$ and let $T_0 = \{e_1, \dots, e_{(n'-1)/2}\}$ be a subtree of T on $n' \geq (1 - \eta)n$ vertices. Let H be a 3-graph on n vertices, and ϕ be an embedding $\phi : V(T_0) \rightarrow V(H)$. Suppose \mathcal{A} is a family of vertex-disjoint pairs of Δ -stars such that every tuple in \mathcal{A} is covered by ϕ and $|A_\Delta(w_1, w_2, w_3) \cap \mathcal{A}| \geq \alpha n$ for every triple (w_1, w_2, w_3) of distinct vertices of H . Then there is an embedding of T into H .*

The proof of the following covering lemma is analogous to [6, Lemma 9.7].

Covering lemma. *Let $1/n \ll \beta \ll \nu \ll \gamma, 1/\Delta$. Let H be a 3-graph on n vertices with minimum degree $(\frac{1}{2} + \gamma) \binom{n}{2}$ and let T be a loose tree on νn vertices with maximum degree*

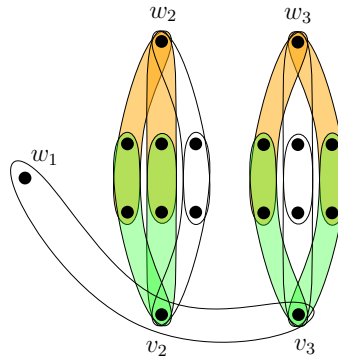


Figure 2: A pair (S_{v_2}, S_{v_3}) of 3-stars which is absorbing for (w_1, w_2, w_3) and covered by an embedding ϕ . Images of edges under ϕ are shown in green. The crucial property of this structure is that the two green stars in ϕ can be switched for the two orange stars plus an extra edge at w_1 , thus extending the embedding.

Δ . Let \mathcal{A} be a set of at most βn pairwise vertex-disjoint absorbing star-pairs in H . Then there is an embedding $\phi : V(T) \rightarrow V(H)$ such that every absorbing tuple in \mathcal{A} is covered by ϕ .

In the following lemma we show that a bounded-degree tree T of size almost n can be embedded in our host graph H . To prove this, we first apply the weak regularity lemma to H to obtain an ε -regular partition of H . The cluster graph inherits the minimum degree of H , and so by the main result in [7] it contains a tight Hamilton cycle $\mathcal{C} = (V_1, \dots, V_t)$. The properties of the regular partition give an embedding of T as long as we can produce what we call a *valid assignment* $a : V(T) \rightarrow [t]$ of its vertices to the clusters of \mathcal{C} . A valid assignment satisfies the following two properties:

- the total number of vertices assigned to each V_j does not exceed $(1 - \eta)|V_j|$, where $\eta \gg \varepsilon$,
- all edges of T are assigned to edges of \mathcal{C} .

Our key idea for finding a valid assignment is to break down the almost-spanning tree into linear-sized pieces, assign these pieces to different edges of \mathcal{C} , and then ‘wrap’ around the tight Hamilton cycle \mathcal{C} to connect the pieces to each other. When assigning a piece of our tree to an edge of \mathcal{C} , we always make sure to leave approximately the same number of vertices unused in each cluster of that edge of \mathcal{C} , so that there is always at least one edge with the capacity to assign an extra piece to it. Since \mathcal{C} has constantly many edges, wrapping around it to connect the pieces only uses up constantly many vertices and so does not interfere with our balance invariant.

Approximate embedding lemma. Let $1/n \ll \eta \ll \gamma, 1/\Delta$, and let H be a 3-graph on n vertices with $\delta_1(H) \geq (\frac{5}{9} + \gamma) \binom{n}{2}$. Let T be a loose tree of maximum degree Δ on at least $(1 - \eta)n$ vertices. Then for every $x \in V(T)$ and $z \in V(H)$, there exists an embedding of T into H that maps x to z .

We are now ready to put these five lemmas together to prove our main result.

Proof of Theorem 1.1. Let $1/n \ll \eta < \alpha \ll \beta \ll \nu \ll \gamma, 1/\Delta$, where n is odd.

We first apply Absorbing set lemma to get a set \mathcal{A} of at most βn pairwise vertex-disjoint pairs of stars, such that for every triple (w_1, w_2, w_3) of vertices in H we have that $|A_\Delta(w_1, w_2, w_3) \cap \mathcal{A}| \geq \alpha n$.

Next, root T arbitrarily at some vertex r and find a subtree $T_x \subset T$ of size $\nu n \leq v(T_x) \leq 2\Delta\nu n$. This can be done by setting $x := r$ and, until x has a child y whose subtree has at least νn vertices, set $x := y$. At some point this process reaches a vertex x whose subtree T_x has at least νn vertices, but all its children's subtrees have fewer than νn vertices, implying that $v(T_x) \leq 2\Delta\nu n$. Let $\nu' := v(T_x)/n$ and apply Covering lemma with $\nu := \nu'$ and $T := T_x$ to find an embedding $\phi_1 : V(T_x) \rightarrow V(H)$ such that every pair of stars in \mathcal{A} is covered by ϕ_1 . Denote $\phi_1(x) = z$.

Now let $H_1 := (H \setminus \phi_1(T_x)) \cup \{z\}$ and note that $\delta_1(H_1) \geq \left(\frac{5}{9} + \frac{\gamma}{2}\right) \binom{|H_1|}{2}$. Let $T_1 := (T \setminus T_x) \cup \{x\}$ and root T_1 at x . Remove leaf edges from T_1 repeatedly to get T_2 such that $v(T_1) - v(T_2) = \eta|H_1|$. Apply Approximate embedding lemma with $H := H_1$ and $T := T_2$ to find an embedding ϕ_2 of T_2 into H_1 with $\phi_2(x) = z$.

Finally, let $T_3 := T_x \cup T_2$ and note that $v(T_3) = n - \eta|H_1| \geq (1 - \eta)n$. Combine ϕ_1 and ϕ_2 into an embedding ϕ_3 of T_3 into H , which can be done since $\phi_1(T_x) \cap \phi_2(T_2) = \phi_1(x) = \phi_2(x)$. Then the tree T_3 , the embedding ϕ_3 , and the set of absorbing tuples \mathcal{A} satisfy the conditions of Absorbing lemma, which we can apply to get an embedding of T in H . \square

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