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Product-free sets in the free group

(Extended abstract)

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Abstract

We prove that product-free sets of the free group over a finite alphabet have maximum density 1/2 with respect to the natural measure that assigns total weight one to each set of irreducible words of a given length. This confirms a conjecture of Leader, Letzter, Narayanan and Walters. In more general terms, we actually prove that strongly k-product-free sets have maximum density $1/k$ in terms of the said measure. The bounds are tight.

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1 Introduction

A subset S of a group is said to be *product-free* if there do not exist $x, y, z \in S$ (not necessarily distinct) such that $z = x \cdot y$. Much has been studied about product-free subsets of finite groups, particularly so in the abelian case, where they are usually called *sum-free subsets* (see, for example, the survey by Tao and Vu [8]). The study of product–free subsets in nonabelian groups can be traced back to Babai and Sós [1], see the survey by Kedlaya [3]. Interest on the problem was prompted by the seminal work of Gowers on quasirandom groups [2].

The study of *product-free* sets in discrete infinite structures is more recent. As a first approach to the study of the infinite case, Leader, Letzter, Narayanan, and Walters in [4] proved that product-free subsets of the free *semigroup* on the finite alphabet $\mathcal{A}$ have maximum density 1/2 with respect to the measure that assigns a weight of $|\mathcal{A}|^{-n}$ to every

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word of length n. They conjectured that this is also true for the analogous measure on the free group. The main purpose of the present paper is to provide a proof of this conjecture.

More precisely, let us write $\mathcal{F}$ for the free group over a finite alphabet $\mathcal{A}$. For $n \geq 1$ and $A \subseteq \mathcal{F}$, we write $A(n) = \{w \in A \colon |w| = n\}$ for the set of elements of A whose reduced words have length n, and $A_{\leq n}$ for those that have length smaller or equal than n. We define a measure μ on $\mathcal{F}$ such that $\mu(\{w\}) = 1/|\mathcal{F}(|w|)|$ for all $w \in \mathcal{F}$, so that every layer of words of a given length has the same total weight. Finally, we write $\bar{d}(A) = \limsup_{n\to\infty} \frac{\mu(A_{\leq n})}{\mu(\mathcal{F}_{\leq n})}$ for the upper asymptotic density of A. Our main result can be phrased as follows.

Theorem 1. *Let $S \subseteq \mathcal{F}$ be a product-free subset. Then*

$$\bar{d}(S) \leq \frac{1}{2}. \tag{1}$$

We actually study a generalisation of this result. Following the notion by Łuczak and Schoen [6] we call a subset S of a semigroup *k-product-free* for $k \geq 2$ if there are no $x_1, \ldots, x_k, y \in S$ such that $x_1 \ldots x_k = y$ and, furthermore, we call S *strongly* k-product-free if it is l-product-free for all l with $2 \leq l \leq k$. We are able to prove the following.

Theorem 2. *Let $S \subseteq \mathcal{F}$ be a strongly k-product-free subset for $k \geq 2$. Then*

$$\bar{d}(S) \leq \frac{1}{k}. \tag{2}$$

Fixing an arbitrary $x \in \mathcal{A}$, the natural example of the set $S \subseteq \mathcal{F}$ consisting of words such that the number of x minus the number of x^{-1} in its reduced form is congruent to 1 modulo k, which is strongly k–product–free and has upper asymptotic density $\bar{d}(S) = 1/k$, shows that the upper bound in Theorem 2 is best possible.

We split the proof of Theorem 2 in two steps. The first step consists in reducing the problem to an analogous one over a particular semigroup. The second step consists in proving the theorem over this semigroup, where we may use similar arguments to those of [4]. However, their argument seems to break when considering strongly k-product-free subsets for $k > 3$. We avoid the obstruction we encounter by restricting our analysis to a subsemigroup where S is at least as dense and pseudorandom in a certain weak sense. This is achieved via a density increment argument.

To conclude, we also want to remark that the statement of Theorem 2 also holds in the free semigroup, which actually was the model where we first worked out the proof, and it is a generalisation of the main theorem in [4].

Theorem 3. *For any finite alphabet $\mathcal{A}$, a strongly k-product-free of the free semigroup over $\mathcal{A}$ has upper asymptotic density at most $1/k$.*

Details of the proofs not included here can be found in [7].

2 Reduction to a semigroup

We write $\mathcal{F}^{xy} \subseteq \mathcal{F}$ for the subset of words that begin with x and end in y (with $x, y \in \mathcal{A} \cup \mathcal{A}^{-1}$). The first step of the proof consists in reducing the proof of Theorem 2 to an analogous statement over $\mathcal{F}^{xy}$ with $x \neq y^{-1}$. This ambient space has the advantage of having no cancellation when multiplying, so it is much closer to the case of the free semigroup, and we are then be able to use ideas similar in spirit to those of [4].

For a given family $\mathcal{H} \subseteq \mathcal{F}$ and a subset $A \subseteq \mathcal{H}$, we define the *relative upper density* of A as

$$\bar{d}_{\mathcal{H}}(A) = \limsup_{n \to \infty} \frac{\mu(A_{\leq n})}{\mu(\mathcal{H}_{\leq n})}.$$

The analogous result to Theorem 2 then reads as follows.

Proposition 1. *Let $S \subseteq \mathcal{F}^{xy}$ be a strongly k-product-free with $k \geq 2$ and $x \neq y^{-1}$. Then*

$$\bar{d}_{\mathcal{F}^{xy}}(S) \leq \frac{1}{k}.$$

The proof of Theorem 2 starts by splitting the elements of $S \subset \mathcal{F}$ according to their initial and final letter. If these are not opposite, then we may apply Proposition 1. If they are opposite, the crucial observation is that we are conjugating by a certain letter, and hence the property of being product-free is preserved. We may then split again according to the second and next to last letters. Iterating this argument, the increase in density over $1/k$ must come from words of the form $w\alpha w^{-1}$ for w of large length, which have exponentially low total mass.

3 Proof over a semigroup

Let us give a brief overview of the proof of Proposition 1. Assume fixed $x, y \in \mathcal{A} \cup \mathcal{A}^{-1}$ such that $x \neq y^{-1}$ and write $\mathcal{G} = \mathcal{F}^{xy} \subseteq \mathcal{F}$. Also write $K = \frac{2|\mathcal{A}|}{2|\mathcal{A}|-1}$ for a constant that appears in several arguments, due to the fact that $|\mathcal{F}(i)||\mathcal{F}(j)| = \frac{|\mathcal{F}(i+j)|}{K}$.

We first prove a version of Proposition 1 that depends on the construction of certain subsets of S with appropriate properties. We say $\mathcal{H} \subseteq \mathcal{F}$ is *dense* if $\mu(\mathcal{H}_{\leq n}) > \delta\mu(\mathcal{F}_{\leq n}) > 0$ for a fixed δ and all large enough n. Furthermore, it is a *subsemigroup* if $\mathcal{H} \cdot \mathcal{H} \subseteq \mathcal{H}$, i.e. $\alpha\beta \in \mathcal{H}$ for all $\alpha, \beta \in \mathcal{H}$. Finally, a subset $W \subseteq \mathcal{H}$ has *unique products* in $\mathcal{H}$ if the map

$$\begin{aligned}(W, \mathcal{H}) &\to \mathcal{H}\\ (w, h) &\mapsto w \cdot h\end{aligned}$$

is injective.

For W and $\mathcal{H} \subseteq \mathcal{G}$ satisfying the above properties, we prove a version of Proposition 1 conditional on $\mu(W)$ being close to K.

Lemma 1. *Let $\mathcal{H} \subseteq \mathcal{G}$ be a dense subsemigroup. For any strongly k-product free set $S \subseteq \mathcal{H}$ and finite subset $W \subseteq S$ with unique products it holds that*

$$\bar{d}_{\mathcal{H}}(S) \leq \frac{1}{1 + \frac{\mu(W)}{K} + \cdots + \left(\frac{\mu(W)}{K}\right)^{k-1}}$$

The previous lemma is proved, essentially, through a counting argument. Since S is k-product free, the sets $W^i \cdot S$ will all be disjoint, and hence they cannot be too large. The properties of W allow us to lower bound the size of $W \cdot S$ in terms of the sizes of W and S. We phrase the argument in a probabilistic manner.

In order to prove Proposition 1, we need to build a subset W large enough to apply the previous lemma. We do so in two different ways. We first present a proof for the case $2 \leq k \leq 3$, where we can use a similar argument to the one in [4]. In this case, we may exploit the fact that S is product-free to construct W in a relatively straightforward manner.

The straightforward argument fails for $k > 3$. Thus, we must find another way of building the subset $W \subseteq S$ necessary to apply Lemma 1. To do so, instead of finding such a set directly in $\mathcal{G}$, we find a subset of $\mathcal{G}$ where S is regularly distributed, where the existence of $W \subseteq S$ as we are interested in is much easier to prove and does not depend on S being product-free.

Concretely, we are interested in studying S when we restrict ourselves to words which are divisible in $\mathcal{G}$ by a given factor. For a given $w \in \mathcal{G}$, we write $w\mathcal{G} \subseteq \mathcal{G}$ for the set of words belonging to $\mathcal{G}$ which may be written as $w\alpha$ for $\alpha \in \mathcal{G}$. We then define the following pseudorandomness condition, which measures whether S is evenly distributed when restricted to such sets.

Definition 1. Given $w \in \mathcal{G}$, a subset $S \subset w\mathcal{G}$ is ε-regular in $w\mathcal{G}$ if

$$\bar{d}_{ww'\mathcal{G}}((S \cap (ww'\mathcal{G})) < \bar{d}_{w\mathcal{G}}(S) + \varepsilon \tag{3}$$

for all words $w' \in \mathcal{G}$.

We then prove the analogous statement to Theorem 2 under pseudorandomness assumptions. In particular, the following Lemma implies Theorem 2 when S is ε-regular for all $\varepsilon > 0$.

Lemma 2. *Let $S \subset w\mathcal{G}$ be a strongly k-product free set that is ε-regular in $w\mathcal{G}$, with $w \in \mathcal{G}$, and let $d = \bar{d}_{w\mathcal{G}}(S)$ be its relative upper density. Then*

$$d\left(1 + \frac{d}{d + 2\varepsilon} + \cdots + \left(\frac{d}{d + 2\varepsilon}\right)^k\right) \leq 1, \tag{4}$$

Finally, we use a density-increment strategy, where failure of pseudorandomness implies an increase in density, to find w such that $S \cap w\mathcal{G}$ is pseudorandom, and apply the previous lemma in this setting.

4 Final remarks

The proof of Theorem 3 is done following the arguments from the previous section, by only replacing the role played by $\mathcal{F}^{xy}$ for $\boldsymbol{F}_{\mathcal{A}}$, and by replacing K by 1. It is also worth noting that the results in [4] concern the upper Banach density of sum-free subsets, which gives slightly stronger results, since the upper Banach density is an upper bound for the upper asymptotic density we consider. For the sake of simplicity, we have not attempted to write down our results for this case, although all arguments go through.

Finally, it would also be interesting to consider the case of k-product-free sets. To state the natural conjecture for this case, define ρ as

$$\rho(l) = \min\left(\{l \in \mathbb{N} \colon l \nmid k - 1\}\right).$$

Then we believe the following to be true.

Conjecture 1. *Let $S \subseteq \mathcal{F}$ be a k-product-free subset for $k \geq 2$. Then*

$$\bar{d}(S) \leq \frac{1}{\rho(k)} \tag{5}$$

This is analogous to a result of Łuczak and Schoen [5], which proves the corresponding statement over the integers.

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