Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

Vizing's Interchange Conjecture holds for simple graphs

(Extended abstract)

Jonathan Narboni*

Abstract

In 1964 Vizing proved that starting from any k-edge-coloring of a graph G one can reach, using only Kempe swaps, a $(\Delta+1)$-edge-coloring of G where Δ is the maximum degree of G. One year later he conjectured that one can also reach a Δ-edge-coloring of G if there exists one. Bonamy *et. al* proved that the conjecture is true for the case of triangle-free graphs. In this paper we prove the conjecture for all simple graphs.

DOI: https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-102

1 Introduction

In 1964 Vizing [6] proved that the chromatic index of a graph G (*i.e.*the minimum number of colors needed to properly colors the edges of G), denoted by $\chi'(G)$, is at most $\Delta(G)+1$ colors, where $\Delta(G)$ is the maximum degree of G.

Theorem 1. *Any simple graph G satisfy $\chi'(G) \leqslant \Delta(G)+1$.*

The proof heavily relies on the use of *Kempe changes*. Kempe changes were introduced by Kempe in his unsuccessful attempt to prove the 4-color theorem, but it turns out that this concept became one of the most fruitful tool in graph coloring. Throughout this paper, we will mostly focus on simple graphs (with no multiple edges), and thus, except stated explicitly, the graphs we consider are never multigraphs.

Moreover, we only consider proper edge-colorings, and so we will only write colorings to denote proper edge-colorings. Given a graph G and a coloring β, a Kempe chains C is

*The author is supported by National Science Center of Poland grant 2019/34/E/ST6/00443.

a maximal bichromatic component (Kempe chains were invented in the context of vertex-coloring, but the principle remains the same for edge-coloring). Applying a *Kempe swap* (or Kempe change) on C consists in switching the two colors in this component. Since C is maximal, the coloring obtained after the swap is guaranteed to be a proper coloring, and if C is not the unique maximal bichromatic component containing these two colors, the coloring obtained after the swap is a coloring different from β, as the partition of the edges is different.

The Kempe swaps induce an equivalence relation on the set of colorings of a graph G; two colorings β and β' are equivalent if one can find a sequence of Kempe swaps to transform β into β'. In 1964, Vizing actually proved a stronger statement, he proved that any k-coloring of a graph G (with $k > \Delta(G)$) is equivalent to a $(\Delta(G) + 1)$-coloring of G.

Theorem 2. *Let G be a graph and β a k-coloring of G (with $k > \Delta(G)$). Then there exists a $(\Delta(G) + 1)$-coloring β' equivalent to β.*

Note that some graphs only need Δ colors to be properly colored and thus the existence of an equivalent optimal coloring is not guaranteed by the theorem. One year later, Vizing [7] proved that this result is generalizable to multigraphs, and states the following conjecture (see [5] for more on the history of this conjecture):

Conjecture 3 (Vizing's Interchange Conjecture)**.** *For any multigraph G and for any k-coloring β of G, is there always an optimal coloring equivalent to β ?*

Note that both in Theorem 1 and in Conjecture 3 we do not have the choice in the target coloring. If we can choose a specific target optimal coloring, then the question can be reformulated as a reconfiguration question.

Question 4. For any multigraph G and for any k-coloring β, is any optimal coloring always equivalent to β ?

If true, Question 4 would imply the following conjecture, as it suffices to take the optimal target coloring as an intermediate between the two non-optimal colorings.

Conjecture 5. *Let G be a graph and let $k > \chi'(G)$. Then any two k-colorings are equivalent.*

Mohar proved the case where we have at least two more colors than the optimal [4].

Theorem 6 ([4])**.** *Let G be a graph. Then all $(\chi'(G) + 2)$-colorings are equivalent.*

When considering case where we only have more color than the optimal ($k = \chi'(G)+1$), McDonald & al. proved Conjecture 5 for graphs with maximum degree 3 [3], Asratian and Casselgren proved it for graphs with maximum degree 4 [1], and Bonamy & al. proved that the conjecture is true for triangle-free graphs. In this paper, we prove that the conjecture is true for all graphs.

Theorem 7. *Let G be a graph. Then all its $(\chi'(G) + 1)$-colorings are Kempe-equivalent.*

Theorem 7 is a direct consequence of the following Lemma which is the main result of this paper.

Lemma 8. *Let G be a graph. Then any $(\chi'(G)+1)$-coloring of G is equivalent to any $\chi'(G)$-coloring of G.*

Note that Lemma 8 together with Theorem 1 directly imply Conjecture 3 for (simple) graphs. Indeed, if G is a graph and β is a k-coloring of G with $k > \chi'(G)$, by Theorem 1, the coloring β is equivalent to a $(\Delta(G)+1)$-coloring of G, and thus it is equivalent to a $(\chi'(G)+1)$-coloring β' of G. Apply Lemma 8 to this coloring β' gives an optimal coloring equivalent to β, and thus proves Conjecture 3.

2 General setting of the proof

The proof inherits the technical setup of [2], in this section, we introduce this setting, and give the general outline of the proof of the main result. The majority of the proofs have been deferred to the appendices.

2.1 Reduction to $\chi'(G)$-regular graphs

The general setting of the proof follows that of [2] which itself follows that of [3] and of [1]. We first show that we can reduce the problem to the class of regular graphs.

Lemma 9. *Let G be a graph. Then there exists a graph $\chi'(G)$-regular graph G' such that:*

- *G is an induced subgraph of G,*
- *any $(\chi'(G)+1)$-coloring of G can be completed into a $(\chi'(G)+1)$-coloring of G', and*
- *if two $(\chi'(G)+1)$-colorings of G' are equivalent, then their restrictions to G are also equivalent.*

Note that colorings in regular graphs are easier to handle due to the following two properties:

- for any $(\Delta(G))$-coloring of a $\chi'(G)$-regular graph G, every vertex v is incident to exactly one edge of each color, and each color class is a perfect matching, and
- for any $(\Delta(G)+1)$-coloring α of a $\chi'(G)$-regular graph G, every vertex v is incident to all but one color, we call this color the *missing color* at v, and denote it by $m_\alpha(v)$ (we often drop the α when the coloring is clear from the context).

From now on, in the rest of the paper, we only consider χ'-regular graphs.

2.2 The good, the bad, and the ugly

The general approach to Lemma 8 is an induction on the chromatic index. Given a graph G, a $\Delta(G)$-coloring α and a $(\Delta(G)+1)$-coloring β, our goal is to find a sequence of Kempe swaps to transform β into α. To do so, we will "allign" α and β using the following lemma.

Lemma 10. *Let G be a regular graph, β be a $(\Delta(G)+1)$-coloring of G, α be a $\Delta(G)$-coloring of G, and let c be a color of α. Then the coloring β is equivalent to a $(\Delta(G)+1)$-coloring β' where for any edge e we have $\beta'(e) = c \Leftrightarrow \alpha(e) = c$.*

For any color class c in α, say $c = 1$, the edges colored 1 induce a perfect matching M in G. So once we obtain the coloring β' using Lemma 10 for the color 1, we can then proceed by induction of $G' = G \setminus M$. Remark that $\chi'(G') = \chi'(G) - 1$, and that the restrictions of α and β' to G' only use $\Delta(G) - 1$, and $\Delta(G)$ colors respectively since the color 1 is not used anymore.

Given a $(\Delta(G)+1)$-coloring β of G and a color, say the color 1. Let M be the perfect matching induced by the edges colored 1. We can partition the edges of G into three sets, an edge e is called:

- *good*, if $e \in M$ and $\beta(e) = 1$,
- *bad*, if $e \in M$ and $\beta(e) \neq 1$, and
- *ugly*, if $e \notin M$ and $\beta(e) = 1$.

A vertex missing the color 1 is called a *free vertex*. Toward contradiction, we assume that β is not equivalent to α, and we consider a $(\Delta(G)+1)$-coloring β' equivalent to β which minimizes the number of ugly edges among the colorings equivalent to β that minimize the number of bad edges, we call such a coloring *minimal*. Observe that in a minimal coloring there exists a bad edge. Thus, if we can find a coloring β'' equivalent to β' where the number of bad edges is strictly lower than in β', or with the same number of bad edges, and strictly fewer ugly edges, we get a contradiction. In a minimal coloring, we first have the following property.

Lemma 11. *In a minimal coloring, there exists a bad edge adjacent to an ugly edge and incident with a free vertex.*

2.3 Fan-like tools

In his proof of Theorem 1, Vizing introduced a technical tool to apply Kempe swaps on a coloring in very controlled way: *Vizing fans*. To define them, we will use an auxiliary digraph. Vizing did not use a digraph to define the fans, but this definition will prove to be suitable for our method of proof. Given a graph G, a $(\Delta(G)+1)$-coloring β of G and a vertex v, the directed graph D_v is defined as follows:

- the vertex set of D_v is the set of edges incident with v, and

- we put an arc between two vertices vv_1 and vv_2 of D_v, if the edge vv_2 is colored with the missing color at v_1.

The *fan around* v *starting at the edge* e, denoted by $X_v(e)$, is the maximal sequence of vertices of D_v reachable from the edge e. It is sometimes more convenient to speak about the color of the starting edge of a fan: If c is a color, then $X_v(c)$ denotes the fan around v starting at the edge colored c incident with v. Note that since the graph G is $\chi'(G)$-regular, each vertex misses exactly one color, and thus, in the digraph D_v, each vertex has outdegree at most 1. Hence a fan $\mathcal{X}$ is well-defined and we only have three possibilities for the fan $\mathcal{X}$:

- $\mathcal{X}$ is a path,
- $\mathcal{X}$ is a cycle, or
- $\mathcal{X}$ is a *comet* (*i.e.*, a path with an additional arc between the last vertex of the path and an internal vertex of the path).

If $\mathcal{X} = (vv_1, \cdots, vv_k)$ is a fan, v is called the central vertex of the fan, and vv_1 and vv_k are respectively called the first and the last edge of the fan (similarly, v_1 and v_k are the first and last vertex of $\mathcal{X}$ respectively). For any fan $\mathcal{V} = (vv_1, \cdots, vv_k)$ in a coloring β, $V(\mathcal{V})$ denotes the set of vertices $\{v_1, \cdots v_k\}$, and $E(\mathcal{V})$ denotes the set of edges $\{vv_1, \cdots vv_k\}$. We denote by $\beta(\mathcal{V})$ the set of colors involved in $\mathcal{V}$ (*i.e.* $\beta(\mathcal{V}) = \beta(E(\mathcal{V})) \cup m(V(\mathcal{V})) \cup m(v)$); if $\mathcal{V}$ involves the color c, $M(X, c)$ denotes the vertex of $V(\mathcal{V})$ missing the color c if this vertex is unique.

Given a $(\Delta(G)+1)$-coloring β of G, and fan $\mathcal{X} = (vv_1, \cdots, vv_k)$ which is a cycle around a vertex v, where each vertex v_i misses the color i (and so each edge vv_i is colored $(i-1)$), we can define the coloring $\beta' = X^{-1}(\beta)$ as follows:

- for any edge vv_i not in $\mathcal{X}$, $\beta'(vv_i) = \beta(vv_i)$, and
- for any edge vv_i in $\mathcal{X}$, $\beta'(vv_i) = i$ and $m(v_i) = i - 1$

The coloring $\mathcal{X}^{-1}(\beta)$ is called the *invert* of $\mathcal{X}$, and we say that X is *invertible* if $\mathcal{X}$ and $\mathcal{X}^{-1}(\beta)$ are equivalent. If the cycle $\mathcal{X}$ is invertible, inverting $\mathcal{X}$ in β means applying a sequence of Kempe swaps to obtain $\mathcal{X}^{-1}(\beta)$ from the coloring β. In this paper, we prove that in any coloring, any cycle is invertible. This is the key ingredient of the proof of Lemma 10.

Lemma 12. *In any* $(\chi'(G)+1)$*-coloring of a* $\chi'(G)$*-regular graph* G*, any cycle is invertible.*

The proof of Lemma 12 is an induction on the size of the cycles. Towards contradiction, assume that there exist non-invertible cycles. A *minimum* cycle $\mathcal{V}$ is a non-invertible cycle of minimum size (*i.e.*, in any coloring, any smaller cycle is invertible).

A cycle of size 2 is clearly invertible as it only consists of a single Kempe chain composed of exactly two edges: to invert the cycle, it suffices to apply a Kempe swap on this component; so the size of a minimum cycle is at least 3. To prove the lemma, we need the two following results.

Lemma 13. *Let $\mathcal{V}$ be a minimum cycle around a vertex v. For any color c different from $m(v)$, the fan $X_v(c)$ is a cycle.*

Lemma 14. *Let $\mathcal{V}$ be a minimum cycle around a vertex v, and $\mathcal{X}$ and $\mathcal{Y}$ be two cycles around v. For any pair of vertices (z, z') in $(\mathcal{V} \cup \mathcal{X} \cup \mathcal{Y})^2$, the fan $\mathcal{Z} = X_z(m(z'))$ is a cycle containing z'.*

Proof of Lemma 12. To prove the Lemma, we prove that the graph G only consists of an even clique where each vertex misses a different color. This is a contradiction since in any $(\Delta(G) + 1)$-coloring of an even clique, for any color c, the number of vertices missing the color c is always even. By Lemma 13, all the fans around v are cycles, so each neighbor of v misses a different color. Moreover, by Lemma 14, there is an edge between each pair of neighbors of v, so $G = N[v] = K_{\Delta(G)+1}$. By construction, G is $\Delta(G)$-colorable, so G is an even clique and each vertex misses a different color as desired. □

The key ingredient of the proof of Lemma 13 is the notion of *entangleness* between two fans. Let $\mathcal{X}$ and $\mathcal{X}'$ be two fans, the fans $\mathcal{X}$ and $\mathcal{X}'$ are called *entangled* if for any color $c \in \beta(\mathcal{X}) \cap \beta(\mathcal{X}')$, we have $M(\mathcal{X}, c) = M(\mathcal{X}', c)$. Thus if two fans that are entangled and share a color, then their central vertices have a common neighbor. Note that if G is a triangle free graph, and uv is an edge of G, then a fan around v cannot be entangled with a fan around u if these two fans share a color. To prove Lemma 13 we need the two following lemmas.

Lemma 15. *Let $\mathcal{V}$ be a minimum cycle in a coloring β and let u and u' be two vertices of $\mathcal{V}$. Then fan $\mathcal{U} = X_u(m(u')) = (uu_1, \cdots, uu_l)$ is a cycle entangled with $\mathcal{V}$.*

Lemma 16. *Let $\mathcal{V}$ be a minimum cycle in a coloring β, u and u' be two vertices of $\mathcal{V}$, and $\mathcal{U} = X_u(m(u')) = (uu_1, \cdots, uu_l)$. Then for any $j \leqslant l$, the fan $X_v(\beta(uu_j))$ is a cycle.*

Note that by Lemma 15, we can directly conclude that $N[v]$ is a clique. The proof of Lemma 16 is pretty involved and technical and consists of finding a sequence of Kempe swaps to invert a minimum cycle $\mathcal{V}$ and thus reaching a contradiction. To do so we will use two meta-operations based on Kempe swaps, namely the inversion of fans that are paths, and the inversion of fans that are cycles smaller that the cycle $\mathcal{V}$. The key ingredient of the proof is to consider a whole equivalence class of colorings where the cycle $\mathcal{V}$ is minimum.

Let $X \subseteq E(G) \cup V(G)$, β a coloring and β' a coloring obtained from β by swapping a component C. The component is called X-stable if :

- for any $v \in X$, $m^{\beta}(v) = m^{\beta'}(v)$, and
- for any $e \in X$, $\beta(e) = \beta'(e)$.

In this case, the coloring β' is called X-*identical* to β.

If $S = (C_1, \cdots, C_k)$ is a sequence of swaps to transform a coloring β into a coloring β' where each C_j is a Kempe swap. The sequence S^{-1} is defined as the sequence of swaps $(C_k, \cdots, C_1)$. Such a sequence is called X-stable if each C_j is X-stable. If a sequence

S is X-stable, then the coloring obtained after apply S to β is called X-*equivalent* to β. Note that the notion of X-equivalence is stronger than the notion of X-identity. Since two colorings β and β' may be X-identical but not X-equivalent if there exists a coloring β'' in the sequence between β and β' that is not X-identical to β.

The following observation gives a relation between X-equivalence and $(G \setminus X)$-identity between colorings.

Observation 17. *Let β be a coloring, $X \subseteq V(G) \cup E(G)$, β_1 a coloring X-equivalent to β, and β_2 a coloring $(G \setminus X)$-identical to β_1. Then, there exists a coloring β_3 equivalent to β_2 that is X-identical to β_2 and $(G \setminus X)$-identical to β.*

If $\mathcal{X}$ is a fan, when two colorings are $(V(\mathcal{X}) \cup E(\mathcal{X}))$-identical (respectively $(V(\mathcal{X}) \cup E(\mathcal{X}))$-equivalent), we simply write that the two colorings are $\mathcal{X}$-identical (respectively $\mathcal{X}$-equivalent). Similarly, if two colorings are $((V(G) \cup E(G)) \setminus X)$-identical (respectively $((V(G) \cup E(G)) \setminus X)$-equivalent), we simply write that the two colorings are $(G \setminus X)$-identical (respectively $(G \setminus X)$-equivalent).

Remark that if $\mathcal{V}$ is a cycle in a coloring β, then the coloring $\mathcal{V}^{-1}(\beta)$ is $(G \setminus \mathcal{V})$-identical to β. So from the previous observation we have the following corollary.

Corollary 18. *Let $\mathcal{V}$ be a cycle in a coloring β. If there exists a coloring β' $\mathcal{V}$-equivalent to β where $\mathcal{V}$ is invertible, then $\mathcal{V}$ is invertible in β.*

From the previous corollary, we derive the following observation.

Observation 19. *Let $\mathcal{V}$ be a minimum cycle in coloring β, and β' a coloring $\mathcal{V}$-equivalent to β. Then in the coloring β', the sequence $\mathcal{V}$ is also a minimum cycle such that for any $e \in E(\mathcal{V})$, $\beta(e) = \beta'(e)$, and for any $v \in V(\mathcal{V})$, $m^{\beta}(v) = m^{\beta'}(v)$.*

We simply say that the cycle $\mathcal{V}$ is the same minimum cycle in the coloring β'. And thus it suffices to find a coloring $\mathcal{V}$-equivalent to β where $\mathcal{V}$ is invertible to reach a contradiction.

Acknowledgements. We thank Marthe Bonamy, František Kardoš and Éric Sopena for helpful discussions, and gratefully thank Penny Haxell for pointing out a mistake in an earlier version.

References

[1] Armen S. Asratian and Carl Johan Casselgren. Solution of Vizing's problem on interchanges for the case of graphs with maximum degree 4 and related results. *Journal of Graph Theory*, 82(4):350–373, 2016.

[2] Marthe Bonamy, Oscar Defrain, Tereza Klimošová, Aurélie Lagoutte, and Jonathan Narboni. On Vizing's edge colouring question. *arXiv preprint arXiv:2107.07900*, 2021.

[3] Jessica McDonald, Bojan Mohar, and Diego Scheide. Kempe equivalence of edge-colorings in subcubic and subquartic graphs. *Journal of Graph Theory*, 70(2):226–239, 2012.

[4] Bojan Mohar. Kempe equivalence of colorings. In *Graph Theory in Paris*, pages 287–297. Springer, 2006.

[5] Michael Stiebitz, Diego Scheide, Bjarne Toft, and Lene M Favrholdt. *Graph edge coloring: Vizing's theorem and Goldberg's conjecture*, volume 75. John Wiley & Sons, 2012.

[6] Vadim G. Vizing. On an estimate of the chromatic class of a p-graph. *Discret Analiz*, 3:25–30, 1964.

[7] Vadim G Vizing. The chromatic class of a multigraph. *Cybernetics*, 1(3):32–41, 1965.

[8] Vadim G. Vizing. Some unsolved problems in graph theory. *Russian Mathematical Surveys*, 23(6):125, 1968.