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# Kneser graphs are Hamiltonian*

## (Extended abstract)

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**Abstract**

For integers $k \geq 1$ and $n \geq 2k+1$, the Kneser graph $K(n,k)$ has as vertices all $k$-element subsets of an $n$-element ground set, and an edge between any two disjoint sets. It has been conjectured since the 1970s that all Kneser graphs admit a Hamilton cycle, with one notable exception, namely the Petersen graph $K(5,2)$. This problem received considerable attention in the literature, including a recent solution for the sparsest case $n = 2k+1$. The main contribution of this paper is to prove the conjecture in full generality. We also extend this Hamiltonicity result to all connected generalized Johnson graphs (except the Petersen graph). The generalized Johnson graph $J(n,k,s)$ has as vertices all $k$-element subsets of an $n$-element ground set, and an edge between any two sets whose intersection has size exactly $s$. Clearly, we have $K(n,k) = J(n,k,0)$, i.e., generalized Johnson graph include Kneser graphs as a special case. Our results imply that all known families of vertex-transitive graphs defined by intersecting set systems have a Hamilton cycle, which settles an interesting special case of Lovász' conjecture on Hamilton cycles in vertex-transitive graphs from 1970. Our main technical innovation is to study cycles in Kneser graphs by a kinetic system of multiple gliders that move at different speeds and that interact over time, reminiscent of the gliders in Conway's Game of Life, and to analyze this system combinatorially and via linear algebra.

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## 1 Introduction

For integers $k \geq 1$ and $n \geq 2k+1$, the *Kneser graph* $K(n,k)$ has as vertices all $k$-element subsets of $[n] := \{1,2,\ldots,n\}$, and an edge between any two sets $A$ and $B$ that are disjoint, i.e., $A \cap B = \emptyset$. Kneser graphs were introduced by Lovász [Lov78] in his celebrated proof of Kneser's conjecture. Using the Borsuk-Ulam theorem, he proved that the chromatic number of $K(n,k)$ equals $n-2k+2$. Observe also that the maximum independent set in $K(n,k)$ has size $\binom{n-1}{k-1}$ by the famous Erdős-Ko-Rado [EKR61] theorem. Furthermore, the graph $K(n,k)$ is vertex-transitive, i.e., it 'looks the same' from the point of view of any vertex, and all vertices have degree $\binom{n-k}{k}$. Lastly, note that when $n < ck$, the Kneser graph $K(n,k)$ does not contain cliques of size $c$, whereas it does contain such cliques when $n \geq ck$.

## 2 Hamilton cycles in Kneser graphs

In this work we investigate Hamilton cycles in Kneser graphs, i.e., cycles that visit every vertex exactly once. Kneser graphs have long been conjectured to have a Hamilton cycle, with one notable exception, the Petersen graph $K(5,2)$, which only admits a Hamilton path. As Kneser graphs are vertex-transitive, this is a special case of Lovász' famous conjecture [Lov70], which asserts that every connected vertex-transitive graph admits a Hamilton path. So far, the conjecture for Hamilton cycles in Kneser graphs has been tackled from two angles, namely for sufficiently dense Kneser graphs, and for the sparsest Kneser graphs. From the aforementioned results about the degree and cliques in $K(n,k)$, we see that $K(n,k)$ is relatively dense when $n$ is large w.r.t. $k$, and relatively sparse otherwise. The sparsest case is when $n = 2k+1$, and the graphs $O_k := K(2k+1,k)$ are also known as *odd graphs*. Intuitively, proving Hamiltonicity should be easier for the dense cases, and harder for the sparse cases.

We first recap the known results for dense Kneser graphs. Heinrich and Wallis [HW78] showed that $K(n,k)$ has a Hamilton cycle if $n \geq 2k + k/(\sqrt[k]{2}-1) = (1+o(1))k^2/\ln 2$. This was improved by B. Chen and Lih [CL87], whose results imply that $K(n,k)$ has a Hamilton cycle if $n \geq (1+o(1))k^2/\log k$; see [CI96]. In another breakthrough, Y. Chen [Che00] showed that $K(n,k)$ is Hamiltonian when $n \geq 3k$. A particularly nice and clean proof for the cases where $n = ck$, $c \in \{3,4,\ldots\}$, was obtained by Y. Chen and Füredi [CF02], later extended by Bellmann and Schülke to any $n \geq 4k$ [BS21]. The asymptotically best result known to date, again due to Y. Chen [Che03], is that $K(n,k)$ has a Hamilton cycle if $n \geq (3k+1+\sqrt{5k^2-2k+1})/2 = (1+o(1))2.618\ldots \cdot k$. With the help of computers, Shields and Savage [SS04] found Hamilton cycles in $K(n,k)$ for all $n \leq 27$ (except for the Petersen graph).

We now briefly summarize the Hamiltonicity story of the sparsest Kneser graphs, namely the odd graphs. Note that $O_k = K(2k+1,k)$ has degree $k+1$, which is only logarithmic in the number of vertices. The conjecture that $O_k$ has a Hamilton cycle for all $k \geq 3$ originated in the 1970s, in papers by Meredith and Lloyd [ML72, ML73] and by Biggs [Big79]. Already Balaban [Bal72] exhibited a Hamilton cycle for the cases $k = 3$

and $k = 4$, and Meredith and Lloyd described one for $k = 5$ and $k = 6$. Later, Mather [Mat76] solved the case $k = 7$. Mütze, Nummenpalo and Walczak [MNW21] finally settled the problem for all odd graphs, proving that $O_k$ has a Hamilton cycle for every $k \geq 3$. Already much earlier, Johnson [Joh11] provided an inductive argument that establishes Hamiltonicity of $K(n,k)$ provided that the existence of Hamilton cycles is known for several smaller Kneser graphs. Combining his result with the unconditional results from [MNW21] yields that $K(2k+2^a,k)$ has a Hamilton cycle for all $k \geq 3$ and $a \geq 0$. These results still leave infinitely many open cases, the sparsest one of which is the family $K(2k+3,k)$ for $k \geq 1$.

The main contribution of this paper is to settle the conjecture on Hamilton cycles in Kneser graphs affirmatively in full generality.

**Theorem 1.** *For all $k \geq 1$ and $n \geq 2k+1$, the Kneser graph $K(n,k)$ has a Hamilton cycle, unless it is the Petersen graph, i.e., $(n,k) = (5,2)$.*

More generally, our work settles all known instances of Lovász' conjecture for vertex-transitive graphs defined by intersecting set systems. As we shall see, Kneser graphs are the hardest cases among them to prove, i.e., with the help of Theorem 1 the Hamiltonicity of the more general families of graphs can be settled easily.

## 3 Generalized Johnson graphs

The *generalized Johnson graph* $J(n,k,s)$ has as vertices all $k$-element subsets of $[n]$, and an edge between any two sets $A$ and $B$ that satisfy $|A \cap B| = s$, i.e., the intersection of $A$ and $B$ has size exactly $s$. To ensure that the graph is connected, we assume that $s < k$ and $n \geq 2k - s + \mathbf{1}_{[s=0]}$, where $\mathbf{1}_{[s=0]}$ denotes the indicator function that equals 1 if $s = 0$ and 0 otherwise. Generalized Johnson graphs are sometimes called 'uniform subset graphs' in the literature, and they are also vertex-transitive. Furthermore, by taking complements, we see that $J(n,k,s)$ is isomorphic to $J(n,n-k,n-2k+s)$. Clearly, Kneser graphs are special generalized Johnson graphs obtained for $s = 0$. On the other hand, the graphs obtained for $s = k-1$ are known as (ordinary) *Johnson graphs* $J(n,k) := J(n,k,k-1)$.

Chen and Lih [CL87] conjectured that all graphs $J(n,k,s)$ admit a Hamilton cycle except the Petersen graph $J(5,2,0) = J(5,3,1)$, and this problem was reiterated in Gould's survey [Gou91]. In their original paper, Chen and Lih settled the cases $s \in \{k-1, k-2, k-3\}$. For the Johnson graphs $J(n,k) = J(n,k,k-1)$, strong Hamiltonicity properties are known [TL73, JR94, Kno94].

We generalize Theorem 1 further, by showing that all connected generalized Johnson graphs admit a Hamilton cycle. This resolves Chen and Lih's conjecture affirmatively in full generality.

**Theorem 2.** *For all $k \geq 1$, $0 \leq s < k$, and $n \geq 2k - s + \mathbf{1}_{[s=0]}$ the generalized Johnson graph $J(n,k,s)$ has a Hamilton cycle, unless it is the Petersen graph, i.e., $(n,k,s) \in \{(5,2,0),(5,3,1)\}$.*

# 4 Bipartite Kneser graphs and the middle levels problem

For integers $k \geq 1$ and $n \geq 2k+1$, the *bipartite Kneser graph* $H(n,k)$ has as vertices all $k$-element and $(n-k)$-element subsets of $[n]$, and an edge between any two sets $A$ and $B$ that satisfy $A \subseteq B$. It is easy to see that bipartite Kneser graphs are also vertex-transitive. As $H(n,k)$ is the bipartite double cover of $K(n,k)$, Hamiltonicity of $K(n,k)$ is harder than the Hamiltonicity of $H(n,k)$.

**Lemma 3.** *If $K(n,k)$ admits a Hamilton cycle, then $H(n,k)$ admits a Hamilton cycle or path.*

The sparsest bipartite Kneser graphs $M_k := H(2k+1,k)$ are known as *middle levels graphs*, as they are isomorphic to the subgraph of the $(2k+1)$-dimensional hypercube induced by the middle two levels. The well-known *middle levels conjecture* asserts that $M_k$ has a Hamilton cycle for all $k \geq 1$. This conjecture was raised in the 1980s, settled affirmatively in [Müt16], and a short proof was given in [GMN18]. More generally, all bipartite Kneser graphs $H(n,k)$ were shown to have a Hamilton cycle in [MS17]. Via Lemma 3, our Theorem 1 thus also yields a new alternative proof for the Hamiltonicity of bipartite Kneser graphs. Consequently, our results in this paper settle Lovász' conjecture for all known families of vertex-transitive graphs that are defined by intersecting set systems.

# 5 Proof ideas

It turns out that Theorem 1 can be used to establish Theorem 2 by a simple inductive construction. Consequently, the main work in this paper is to prove Theorem 1. In this extended abstract, we only sketch the main ideas for this proof, for details see [MMN22].

As mentioned before, Mütze, Nummenpalo and Walczak [MNW21] proved that $K(n,k)$ has a Hamilton cycle for $n = 2k+1$ and all $k \geq 3$. Combining this result with Johnson's construction [Joh11] shows that $K(n,k)$ has a Hamilton cycle for $n = 2k+2^a$ and all $k \geq 3$ and $a \geq 0$, in particular for $n = 2k+2$. The techniques developed in this paper work whenever $n \geq 2k+3$, and thus they settle all remaining cases of Theorem 1. Our proof does not work in the cases $n = 2k+1$ and $n = 2k+2$, so the two earlier constructions do not become obsolete.

We follow a two-step approach to construct a Hamilton cycle in $K(n,k)$ for $n \geq 2k+3$. In the first step, we construct a *cycle factor* in the graph, i.e., a collection of disjoint cycles that together visit all vertices. In the second step, we join the cycles of the factor to a single cycle.

## 5.1 Cycle factor construction

The starting point is to consider the characteristic vectors of the vertices of $K(n,k)$. For every $k$-element subset of $[n]$, this is a bitstring of length $n$ with exactly $k$ many 1s at the positions corresponding to the elements of the set. For example, the vertex $\{1,7,9\}$ of $K(9,3)$ is represented by the bitstring 100000101. Clearly, two sets $A$ and $B$ that are vertices of $K(n,k)$ are disjoint if and only if the corresponding bitstrings have no 1s at the same positions.

Our construction of a cycle factor in the Kneser graph $K(n,k)$ uses the following simple rule based on parenthesis matching, a technique pioneered by Greene and Kleitman [GK76]: Given a vertex represented by a bitstring $x$, we interpret the 1s in $x$ as opening brackets and the 0s as closing brackets, and we match closest pairs of opening and closing brackets in the natural way, which will leave some 0s unmatched. This matching is done *cyclically* across the boundary of $x$, i.e., $x$ is considered as a cyclic string. We write $f(x)$ for the vertex obtained from $x$ by complementing all matched bits, leaving the unmatched bits unchanged. For example, $x = 100000101$ is interpreted as $x = ()))))()( = ())$---$()($, where each - denotes an unmatched closing bracket, and then complementing matched bits (the first three and last three in this case) yields the vertex $f(x) = 011000010$. Repeatedly applying $f$ to every vertex partitions the vertices of the Kneser graph into cycles, and we write $C(x) := (x, f(x), f^2(x), \ldots)$ for the cycle containing $x$. For example, for $x$ from before we obtain $C(x) = (100000101, 011000010, 000110001, 100001100, 010000011, \ldots, 000011010)$. Figure 1 shows several more examples of cycles generated by this parenthesis matching rule.

### 5.2 Analysis via gliders

The next key step is to understand the structure of the cycles generated by $f$. We describe the evolution of a bitstring $x$ under repeated applications of $f$ by a kinetic system of multiple gliders that move at different speeds and that interact over time, reminiscent of the gliders in Conway's Game of Life. This physical interpretation and its analysis are one of the main innovations of this paper. Specifically, we view each application of $f$ as one unit of time moving forward. Furthermore, we partition the matched bits of $x$ into groups, and each of these groups is called a *glider*. A glider has a *speed* associated to it, which is given by the number of 1s in its group. For example, in the cycle shown in Figure 1 (a), there is a single matched 1 and the corresponding matched 0, and together these two bits form a glider of speed 1 that moves one step to the right in every time step. Applying $f$ means going down to the next row in the picture, so the time axis points downwards. Similarly, in Figure 1 (b), there are two matched 1s and the corresponding two matched 0s, and together these four bits form a glider of speed 2 that moves two steps to the right in every time step. As we see from these examples, a single glider of speed $v$ simply moves uniformly, following the basic physics law $s(t) = s(0) + v \cdot t$, where $t$ is the time (i.e., the number of applications of $f$) and $s(t)$ is the position of the glider in the bitstring as a function of time (modulo $n$). The situation gets more interesting and complicated when gliders of different speeds interact with each other. For example, in Figure 1 (c), there is one glider of speed 2 and one glider of speed 1. As long as these groups of bits are separated, each glider moves uniformly as before. However, when the speed 2 glider catches up with the speed 1 glider, an overtaking occurs. During an overtaking, the faster glider receives a boost, whereas the slower glider is delayed. This can be captured by augmenting the corresponding equations of motion by introducing an additional term that involves a variable counting the number of overtakings, making the equations non-uniform. For more than two gliders, the equations of motion can be generalized accordingly, by introducing such overtaking counters between any pair of gliders. Nevertheless, as the reader may appreciate from Figure 1 (d), in general it is highly nontrivial to recognize from an arbitrary bitstring $x$ which of its matched bits belong to which glider, and consequently which glider is currently overtaking which other glider. Note that in general the gliders will not be nicely separated, but will be involved in simultaneous interactions, so that the groups of bits forming the gliders will be interleaved in complicated ways.

From the aforementioned physics interpretation we obtain that the number of gliders and their speeds are invariant along each cycle. For example, in Figure 1 (d), every bitstring along this cycle has three gliders of speeds 1, 2 and 3. From the equations of motion we also derive another crucial property, namely that no glider stands still forever, but will move eventually. Note that the speed 1 glider in Figure 1 (d) stands still between time

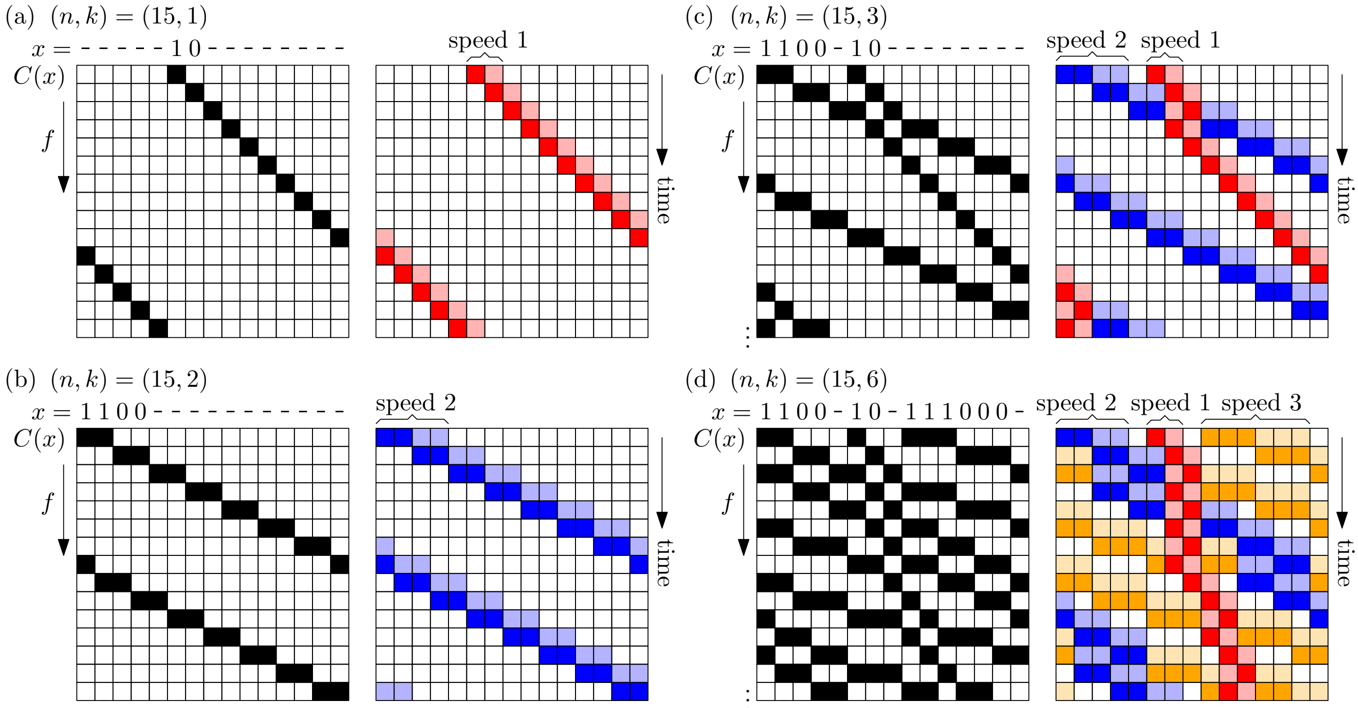


Figure 1: Cycles of our factor in different Kneser graphs $K(n,k)$. The cycles in (a) and (b) are shown completely, whereas in (c) and (d) only the first 15 vertices are shown. Vertices are represented by characteristic vectors, with 1s and 0s shown as black and white squares, resp. In each pair of figures, the right hand side shows the interpretation of certain groups of bits as gliders, and their movement over time. Matched bits belonging to the same glider are colored in the same color, 1-bits filled opaquely, and 0-bits filled transparently. (a) one glider of speed 1; (b) one glider of speed 2; (c) two gliders with speeds 1 and 2 that participate in an overtaking; (d) three gliders of speeds 1, 2 and 3 that participate in multiple overtakings. Animations of these examples are available at [Müt23].

steps 2–8, as during those steps it is overtaken once by the speed 2 glider, and twice by the speed 3 glider (wrapping around the boundary). We establish this fact by linear algebra, by showing that the determinant of the linear systems of equations that governs the gliders' movements is non-singular.

For the reader's entertainment, we programmed an interactive animation of gliders over time, and we encourage experimentation with this code, which can be found at [Müt23].

### 5.3 Gluing the cycles together

To join the cycles of our factor to a single Hamilton cycle, we consider a 4-cycle $D$ that shares two opposite edges with two cycles $C, C'$ from our factor. Clearly, the symmetric difference of the edge sets $(C \cup C')\Delta D$ yields a single cycle on the same vertex set as $C \cup C'$. We may repeatedly apply such gluing operations until all cycles are joined to a single Hamilton cycle. The two main technical obstacles here are: (a) All of the 4-cycles used for the gluing must be edge-disjoint, so that none of the gluings interfere with each other. (b) The gluings must achieve connectivity, i.e., every cycle must be connected to every other cycle via a sequence of gluings. To control the gluing, we consider the speeds of gliders in a bitstring $x$ in non-increasing order. As the sum of speeds equals $k$, this sequence forms a number partition of $k$. To establish (b) we choose gluings that guarantee a lexicographic increase in those number partitions. Specifically, we glue cycles $C(x)$ and $C(y)$ for which the glider speeds in $y$ are obtained from those in $x$ by decreasing the speed of a glider of minimum speed by 1, and by increasing the speed of another glider by 1. This ensures that the number partition of $k$ associated with $y$ is lexicographically larger than that of $x$. Unfortunately, it is not always possible to use gluings that guarantee such immediate lexicographic improvement. In some cases we have to use gluings where a small lexicographic decrease occurs. We then argue that subsequent gluings compensate for this defect such that the overall effect is again a lexicographic improvement. For example, from a vertex with associated number partition $(4, 4)$, the first gluing may lead to a vertex with number partition $(4, 3, 1)$, and the next gluing may lead to $(5, 3)$. While $(4, 4) \to (4, 3, 1)$ is a lexicographic decrease instead of an increase, overall $(4, 4) \to (4, 3, 1) \to (5, 3)$ is a lexicographic increase.

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