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Almost partitioning every 2-edge-coloured complete k-graph into k monochromatic tight cycles

(Extended abstract)

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Abstract

A k-uniform tight cycle is a k-graph with a cyclic order of its vertices such that every k consecutive vertices from an edge. We show that for $k \geq 3$, every red-blue edge-coloured complete k-graph on n vertices contains k vertex-disjoint monochromatic tight cycles that together cover $n - o(n)$ vertices.

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1 Introduction

Monochromatic partitioning is an area of combinatorics that has its origin in a remark of Gerencsér and Gyárfás [9] that any 2-edge-colouring of the complete graph K_n contains a spanning path that consists of a red[1] path followed by a blue path. In particular, every 2-edge-coloured K_n admits a partition of its vertex set into a red path and a blue path. In a subsequent paper, Gyárfás [10] proved the stronger result that every 2-edge-coloured K_n contains a red cycle and a blue cycle that share at most one vertex and together cover

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[1]We assume that the colours in a 2-edge-colouring are always red and blue.

all vertices. Lehel conjectured that every 2-edge-coloured K_n can be partitioned into a red cycle and a blue cycle. Here the empty set, a single vertex, and a single edge are considered to be cycles. Lehel's conjecture was first proved for large enough n by Łuczak, Rödl and Szemerédi [15] using the regularity lemma. Allen [1] later gave a different proof of this that does not use the regularity lemma, thus improving the bound on n. Bessy and Thomassé [3] finally gave a short and elegant proof of Lehel's conjecture for all n.

Here our interest lies in generalisations of Lehel's conjecture to hypergraphs and tight cycles. For related problems in hypergraphs about other types of cycles see [12, 16, 6]. See [11] and [17, Sections 7.4 and 9.7] for surveys on monochromatic partitioning.

For $k \geq 2$, a *k-graph* (or *k-uniform hypergraph*) H is a pair of sets $(V(H), E(H))$ such that $E(H) \subseteq \binom{V(H)}{k}$ (where for a set X, $\binom{X}{k}$ denotes the set of all subsets of X of size k). A *k-uniform tight cycle* is a k-graph with a cyclic order of its vertices such that its edges are exactly the sets of k consecutive vertices in the order. From now on, any set of at most k vertices is also considered a tight cycle.

The problem we want to consider is how to cover almost all vertices of every 2-edge-coloured complete k-graph with as few vertex-disjoint monochromatic tight cycles as possible. For $k = 3$, Bustamante, Hàn and Stein [5] proved that in every 2-edge-coloured complete 3-graph almost all vertices can be partitioned into a red and a blue tight cycle. Subsequently, Garbe, Mycroft, Lang, Lo, and Sanhueza-Matamala [8] showed that in fact there is a partition of all the vertices into two monochromatic tight cycles. Here it is necessary to allow the two monochromatic tight cycles to possibly have the same colour. Indeed for every $k \geq 3$, there exists a complete graph on arbitrarily many vertices that does not admit a partition of its vertices into a red and a blue tight cycle [13, Proposition 1]. In an earlier paper [13], the authors proved that for $k = 4$ it is also possible to almost partition the vertices of every 2-edge-coloured 4-graph into a red and a blue tight cycle. In the same paper, the authors also showed a weaker result for $k = 5$, that 4 vertex-disjoint monochromatic tight cycles suffice to cover almost all vertices of every 2-edge-coloured complete 5-graph. The only bound for general k that was known is given by a result of Bustamante, Corsten, Frankl, Pokrovskiy and Skokan [4]. They showed that every r-edge-coloured complete k-graph on n vertices can be partitioned into $c(r, k)$ monochromatic tight cycles. However, the constant $c(r, k)$ that can be obtained from their proof is very large. Indeed to cover almost all vertices they simply repeatedly find a monochromatic tight cycle using the fact that the Ramsey number for the k-uniform tight cycle on N vertices is linear in N.

Our aim here is to show a reasonable general bound on the number of tight cycles that are needed to almost partition a 2-edge-coloured complete k-graph. Indeed we show that k tight cycles suffice. We remark that this result is probably not tight. The only known lower bound on the number of tight cycles needed is the trivial lower bound of 2.

Theorem 1.1. *For every $\varepsilon > 0$ and $k \geq 3$, there exists an integer n_0 such that every 2-edge-coloured complete k-graph on $n \geq n_0$ vertices contains k vertex-disjoint monochromatic tight cycles covering at least $(1 - \varepsilon)n$ vertices.*

2 Sketch proof of Theorem 1.1

Our proof is based on a hypergraph version of Łuczak's connected matching method (see [14] for the original method). Roughly speaking the idea is as follows. Let K be a 2-edge-coloured complete k-graph on n vertices. We apply the Regular Slice Lemma (a form of the Hypergraph Regularity Lemma that is due to Allen, Böttcher, Cooley and Mycroft [2]) to K^{red}.[2] Since any regularity partition for a k-graph is also a regularity partition for its complement, this gives rise to a reduced k-graph $\mathcal{R}$ that is a 2-edge-coloured almost complete k-graph. So a red edge $i_1 \dots i_k$ in $\mathcal{R}$ means that K^{red} is regular with respect to the corresponding clusters $V_{i_1}, \dots, V_{i_k}$ and at least half the edges of K with one vertex in each V_{i_j}, $j \in [k]$ are red.[3] It can be shown that there is a red tight cycle in K that contains almost all of the vertices in $\bigcup_{j\in[k]} V_{i_j}$. The idea is now to combine a matching M of red edges in $\mathcal{R}$ into an even longer red tight cycle that covers almost all the vertices in the clusters that are covered by M. However, in order for this to work we will need to be able to construct a red tight path[4] that goes from the clusters corresponding to one edge of M to the clusters corresponding to another edge of M. So we require our red matching M in $\mathcal{R}$ to be 'connected' in some sense. To this end, we need the following definitions. A *tight pseudo-walk (of length m from e to e')* in a k-graph H is a sequence of edges $e_1 \dots e_m$ in H such that $|e_i \cap e_{i+1}| = k-1$ (and $(e_1, e_m) = (e, e')$). A k-graph H is *tightly connected* if for every pair of edges $e, e' \in H$, there is a tight pseudo-walk from e to e' in H. A *tight component* in a k-graph H is a maximal tightly connected subgraph of H. Let H be a 2-edge-coloured k-graph. A *red* or *blue tight component* in H is a tight component in H^{red} or in H^{blue}, respectively. A *monochromatic tight component* in H is a red or a blue tight component in H. The hypergraph version of Łuczak's connected matching method that we need now roughly says the following. If $\mathcal{R}$ contains a matching that covers almost all vertices and uses edges from at most k monochromatic tight components, then there exists k monochromatic tight cycles in K that are vertex-disjoint and together cover almost all vertices. The proof of Theorem 1.1 is now reduced to proving that every almost complete 2-edge-coloured k-graph contains a matching that covers almost all vertices and only uses edges from at most k monochromatic tight components.

For a set $S \subseteq V(H)$ with $|S| \le k-1$, we let $N_H(S) = \{S' \in \binom{V(H)}{k-|S|} : S \cup S' \in E(H)\}$ and $d_H(S) = |N_H(S)|$. A k-graph H on n vertices is called (μ, α)-*dense* if, for each $i \in [k-1]$, we have $d_H(S) \ge \mu\binom{n}{k-i}$ for all but at most $\alpha\binom{n}{i}$ sets $S \in \binom{V(H)}{i}$ and $d_H(S) = 0$ for all other sets $S \in \binom{V(H)}{i}$. Note that the reduced graph $\mathcal{R}$ will typically be $(1-\varepsilon, \varepsilon)$-dense.

Our discussion on a hypergraph version of Łuczak's connected matching method is encapsulated in the following corollary from our previous work. When we say that a statement holds for constants a and b with $0 < a \ll b$, we mean that the statement holds provided that a is chosen sufficiently small in terms of b. Moreover, if $1/n$ appears in such

[2] For a 2-edge-coloured k-graph H, we denote by H^{red} and H^{blue} the subgraph induced by the red edges and the subgraph induced by the blue edges of H, respectively.

[3] For $n \in \mathbb{N}$, $[n] = \{1, \dots, n\}$.

[4] A *tight path* is a k-graph with a linear order of its vertices such that every k-consecutive vertices form an edge. Or alternatively a tight path is a k-graph obtained by deleting a single vertex from a tight cycle.

a hierarchy then we implicitly assume that $n \in \mathbb{N}$.

Corollary 2.1 ([13, Corollary 20]). *Let $1/n \ll 1/m \ll \varepsilon \ll \eta \ll \gamma, 1/k, 1/s$ with $k \geq 3$. Suppose that every 2-edge-coloured $(1-\varepsilon, \varepsilon)$-dense k-graph H on m vertices contains a matching in H that covers all but at most ηm vertices of H and only contains edges from at most s monochromatic tight components of H. Then any 2-edge-coloured complete k-graph on n vertices contains s vertex-disjoint monochromatic tight cycles covering at least $(1-\gamma)n$ vertices.*

3 A large matching using edges from few monochromatic tight components

By Corollary 2.1, to prove Theorem 1.1, it suffices to prove the following lemma.

Lemma 3.1. *Let $1/n \ll \varepsilon \ll \eta \ll 1/k \leq 1/2$. Let H be a 2-edge-coloured $(1-\varepsilon, \varepsilon)$-dense k-graph on n vertices. Then there exists a matching in H that covers all but at most ηn vertices of H and only contains edges from at most k monochromatic tight components of H.*

The cases when $k = 3$ is already proved in [5] (in which they showed that a red and a blue tight component suffice). The first step of the proof is to find a red and a blue tight component R and B, respectively, of H such that almost all 2-subsets of $V(H)$ are contained in some edge of $R \cup B$. One then finds a large matching in $R \cup B$. Thus a natural first step of proving Lemma 3.1 is to find a constant number of monochromatic tight components of H, such that almost all $(k-1)$-subsets of $V(H)$ are contained in some edge of these tight components. However this is not possible when $k \geq 4$ as shown by the following example. Let $V_1, \dots, V_\ell$ be an equipartition of a set of n vertices. Consider the 2-edge-coloured complete k-graph on $\bigcup_{i\in[\ell]} V_i$ such that an edge e is red if $|e \cap V_i| > k/2$, and blue otherwise. Observe that there are ℓ blue tight components. Moreover, each $\binom{V_i}{k-1}$ is "covered" by a distinct blue tight component.

Instead, we will enlarge our maximal matching as we choose tight components as follows. Consider a monochromatic tight component F_* in H and let $\mathcal{G}_1 = \{F_*\}$. We say that two monochromatic tight components F_1 and F_2 of H are *adjacent* if they have opposite colours and there are edges $e_1 \in F_1$ and $e_2 \in F_2$ such that $|e_1 \cap e_2| = k-1$. Now for each $i \geq 2$ in turn, let $\mathcal{G}_i$ be the set of monochromatic tight components that are adjacent to a monochromatic tight component in $\mathcal{G}_{i-1}$ and not already in $\bigcup_{j\in[i-1]} \mathcal{G}_j$. Moreover, for each $i \geq 1$, we let $\mathcal{E}(\mathcal{G}_i) = \bigcup_{F\in\mathcal{G}_i} F$, that is, $\mathcal{E}(\mathcal{G}_i)$ is the set of edges that are in some monochromatic tight component $F \in \mathcal{G}_i$. It is easy to see that all edges of H are in $\bigcup_{j\in[2k]} \mathcal{E}(\mathcal{G}_j)$. In fact, if our initial monochromatic tight component F_* spans almost all vertices (such an F^* exists), then almost all edges of H are in $\bigcup_{j\in[k]} \mathcal{E}(\mathcal{G}_j)$. For simplicity, we assume that $H = \bigcup_{j\in[k]} \mathcal{E}(\mathcal{G}_j)$.

We now set $W_0 = V(H)$ and for each $i = 1, \dots, k$ in turn, we let M_i be a maximal matching in $H[W_{i-1}] \cap \bigcup_{j\in[i]} \mathcal{E}(\mathcal{G}_j)$ and $W_i = W_{i-1} \setminus V(M_i)$. Since $\bigcup_{j\in[k]} \mathcal{E}(\mathcal{G}_j) = H$ is $(1-\varepsilon, \varepsilon)$-dense, $\bigcup_{j\in[k]} M_j$ is a maximal matching in H covering almost all vertices of H.

It remains to show that $\bigcup_{j\in[k]} M_j$ is contained in at most k monochromatic tight components. Note that, for each $i \in [k+1]$,

$$H[W_{i-1}] \cap \bigcup_{j\in[i-1]} \mathcal{E}(\mathcal{G}_j) = \varnothing \tag{3.1}$$

by our choices of M_{i-1} and W_{i-1}. Hence $M_i \subseteq H[W_{i-1}] \cap \mathcal{E}(\mathcal{G}_i)$. Therefore, it suffices to show that $H[W_{i-1}] \cap \mathcal{E}(\mathcal{G}_i)$ (and so M_i) consists of edges from one monochromatic tight component.

Suppose for a contradiction that there are two edges e_1 and e_2 in $H[W_{i-1}] \cap \mathcal{E}(\mathcal{G}_i)$ that are in different monochromatic tight components. Suppose further that $|W_{i-1}| \geq \eta n$ (or else $\bigcup_{j\in[i-1]} M_j$ is already an almost perfect matching). In $H[W_{i-1}]$, there exists a tight pseudo-walk P from e_1 to e_2. Recall that $\bigcup_{j\in[i]} \mathcal{E}(\mathcal{G}_j)$ is tightly connected, so $\bigcup_{j\in[i]} \mathcal{E}(\mathcal{G}_j)$ contains a tight pseudo-walk P' from e_2 to e_1. Thus PP' (the concatenation of P and P') is a closed tight pseudo-walk. We then define a nearly triangulated plane graph[5] D such that every vertex of D corresponds to an edge in H, PP' is on the outer face and any walk in D corresponds to a tight pseudo-walk in H. We colour each vertex of D with the same colour of its corresponding edge in H. All edges in $\mathcal{E}(\mathcal{G}_i)$ (including e_1 and e_2) have the same colour, say red. Since e_1 and e_2 are not in the same red tight component, there is no red walk in D from e_1 to e_2. By adapting the proof of Gale [7] of the fact that the Hex game cannot end in a draw, one deduces that H contains a blue tight pseudo-walk P^* from an edge of P to an edge of P'. Since P' is contained in $\bigcup_{j\in[i]} \mathcal{E}(\mathcal{G}_j)$, we have $P^* \subseteq \bigcup_{j\in[i-1]} \mathcal{E}(\mathcal{G}_j)$. Therefore, $\emptyset \neq P \cap P^* \subseteq H[W_{i-1}] \cap \bigcup_{j\in[i-1]} \mathcal{E}(\mathcal{G}_j)$ contradicting (3.1).

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[5] A nearly triangulated plane graph is a plane graph such that all the faces except the outer face are triangles.

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