Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

Cycle Partition of Dense Regular Digraphs and Oriented Graphs

(Extended abstract)

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Abstract

Magnant and Martin [24] conjectured that every d-regular graph on n vertices can be covered by $n/(d+1)$ vertex-disjoint paths. Gruslys and Letzter [11] verified this conjecture in the dense case, even for cycles rather than paths. We prove the analogous result for directed graphs and oriented graphs, that is, for all $\alpha > 0$, there exists $n_0 = n_0(\alpha)$ such that every d-regular digraph on n vertices with $d \geq \alpha n$ can be covered by at most $n/(d+1)$ vertex-disjoint cycles. Moreover if G is an oriented graph, then $n/(2d+1)$ cycles suffice. This also establishes Jackson's long standing conjecture [14] for large n that every d-regular oriented graph on n vertices with $n \leq 4d+1$ is Hamiltonian.

DOI: `https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-099`

1 Introduction

As one of the most intensely studied structures in the graph theory, *a Hamilton cycle* in a (directed) graph is a (directed) cycle that visits every vertex. There are numerous

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results that establish (best-possible) conditions guaranteeing the existence of Hamilton cycles in (directed) graphs. The seminal result of Dirac [5] states that every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ is Hamiltonian. Ghouila-Houri [10] showed the corresponding version in directed graphs (*digraph* for short), that is, every digraph on $n \geq 3$ vertices with minimum semidegree at least $n/2$ (i.e. every vertex has in- and outdegree at least $n/2$) is Hamiltonian. These bounds are tight by taking e.g. the disjoint union of two cliques (a regular extremal example) or a slightly imbalanced complete bipartite graph (an irregular extremal example). Recall that an oriented graph is a digraph that can have at most one edge between each pair of vertices (whereas a digraph can have up to two, one in each direction). For oriented graphs, a more recent result of Keevash, Kühn and Osthus [15] establishes a (tight) minimum degree threshold of $\lceil (3n-4)/8 \rceil$ for Hamiltonicity. In contrast to graphs and digraphs, there are no *regular* extremal examples in the case of oriented graphs. Jackson [14] conjectured in 1981 that regularity actually reduces the degree threshold significantly for oriented graphs:

Conjecture 1.1 (Jackson [14])**.** *For each $d > 2$, every d-regular oriented graph on $n \leq 4d+1$ vertices has a Hamilton cycle.*

The disjoint union of two regular tournaments shows that Jackson's conjecture is best possible.[1] We note that the approximate version of Jackson's conjecture was recently verified by current authors in [23], that is for every $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that every d-regular oriented graph on $n \geq n_0(\varepsilon)$ vertices with $d \geq (1/4+\varepsilon)n$ is Hamiltonian. Here, we verify the exact version for large n.

Theorem 1.2. *There exists an integer n_0 such that every d-regular oriented graph on $n \geq n_0$ vertices with $n \leq 4d+1$ has a Hamilton cycle.*

Generalizing questions about Hamilton cycles, one can consider the question of covering the vertices of a (di)graph by as few vertex-disjoint cycles as possible. Indeed, we prove Theorem 1.2 by showing a more general result about covering regular digraphs and oriented graphs with few vertex-disjoint cycles.

Theorem 1.3. *For all $\alpha > 0$, there exists $n_0 = n_0(\alpha)$ such that every d-regular digraph G on n vertices with $d \geq \alpha n$ can be covered by at most $n/(d+1)$ vertex-disjoint cycles. Moreover if G is an oriented graph, then $n/(2d+1)$ cycles suffice.*

This is best possible by considering the disjoint union of complete digraphs of order $d+1$ for digraphs and the disjoint union of regular tournaments of order $2d+1$ for oriented graphs. Notice that we have $n/(2d+1) < 2$ when $n \leq 4d+1$, so that Theorem 1.3 implies Theorem 1.2. We also note that Theorem 1.3 generalizes the following result of Gruslys and Letzter [11] from regular graphs to regular digraphs and oriented graphs.

Theorem 1.4 (Gruslys and Letzter [11])**.** *For all $\alpha > 0$, there exists $n_0 = n_0(\alpha)$ such that every d-regular graph G on $n \geq n_0$ vertices with $d \geq \alpha n$ can be covered by at most $n/(d+1)$ vertex-disjoint cycles.*

[1] This example works for $n \equiv 2 \pmod 4$. Similar examples can also be constructed when $n \not\equiv 2 \pmod 4$.

Theorem 1.3 implies Theorem 1.4 by making every edge into a directed 2-cycle. Theorem 1.3 also has connections with several well-studied problems in extremal graph theory: here we mention some of them.

1.1 Path Cover

A weaker version of cycle cover is path cover. The *path cover number* $\pi(G)$ of a (di)graph G is the minimum number of vertex-disjoint (directed) paths needed to cover $V(G)$. This was introduced by Ore [25], and he showed that $\pi(G) \leq n - \sigma_2(G)$ holds where $\sigma_2(G)$ denotes the minimum sum of the degrees over all non-adjacent vertices. Magnant and Martin [24] conjectured that *regularity* significantly reduces the upper bound for $\pi(G)$:

Conjecture 1.5 (Magnant and Martin [24])**.** *If G is a d-regular graph on n vertices, then $\pi(G) \leq n/(d+1)$.*

It is known that Conjecture 1.5 holds for small values of d (see [24] for $d \leq 5$ and see [7] for $d = 6$). Han [13] showed that, for dense graphs, it is enough to use $1 + n/(d+1)$ paths to cover almost all vertices. Also, Theorem 1.4 verifies Conjecture 1.5 in the dense case. It is worth noting that the Linear Arboricity Conjecture [2] implies Conjecture 1.5 for odd values of d, and gives $\pi(G) \leq 2n/(d+2)$ for general d (see [7] for a detailed discussion).

For digraphs, the classical result of Gallai and Milgram [9] states that $\pi(G)$ can be bounded above by the size of maximum independent set (and Dilworth's [4] theorem says that the equality holds for the special case of posets). As our Theorem 1.3 generalizes Theorem 1.4 from graphs digraphs and oriented graphs, we believe the following stronger version of Conjecture 1.5 holds, which Theorem 1.3 establishes in the dense case.

Conjecture 1.6. *If G is a d-regular digraph on n vertices, then $\pi(G) \leq n/(d+1)$. Moreover, $\pi(G) \leq n/(2d+1)$ holds if G is oriented.*

Conjecture 1.6 implies Conjecture 1.5 by making every edge into a directed 2-cycle.

1.2 Edge-Disjoint Cycles

In a weaker version of the problem we consider, one is interested in finding *edge-disjoint* cycles whose union covers all the vertices. As a generalization of Dirac's theorem, it is conjectured [6] that if a graph G on n vertices has minimum degree n/k, then $V(G)$ can be covered by $k-1$ edge-disjoint cycles. The case $k = 3$ was also proved in [6]. The conjecture was proved for 2-connected graphs [16], and has been completely resolved in [17]. Later, Balogh, Mousset and Skokan [3] obtained a stability result, showing that every graph on n vertices with minimum degree nearly n/k has a special structure if it does not have $k-1$ edge-disjoint cycles covering all vertices. One can ask the same question for digraphs as a generalization of Ghoulia-Houri's theorem, and Theorem 1.3 answers it affirmatively for regular graphs:

Conjecture 1.7. *If G is a digraph with minimum semi-degree n/k, then $V(G)$ can be covered by $k-1$ edge-disjoint cycles.*

1.3 Extending Perfect Matchings

Gruslys and Letzter [11], as well as proving Theorem 1.4, proved that every large d-regular *bipartite graph* G on n vertices with d linear in n can be covered by at most $n/2d$ vertex-disjoint paths. They mentioned that one should be able to replace paths by cycles. Indeed, as a corollary of Theorem 1.3, the result below shows that those cycles can be found in such a way that they even contain any prescribed perfect matching.

Corollary 1.8. *For all $\alpha > 0$, there exists $n_1 = n_1(\alpha)$ such that, for every d-regular bipartite graph on $n \geq n_1$ vertices with $d \geq \alpha n$, any perfect matching can be extended to a union of at most $n/2d$ vertex-disjoint cycles.*

Note that Corollary 1.8 is tight by considering the disjoint union of $n/2d$ many $K_{d,d}$'s. It also shows that d-regular bipartite graphs on n (sufficiently large) vertices with $d > n/4$ are examples of graphs in which every perfect matching can be extended into a Hamilton cycle. This property is called the PMH-property in [1]. Häggkvist [12] initiated the study of sufficient conditions for the PMH-property (using the name F-Hamiltonian where F is a perfect matching) by showing $\sigma_2(G) \geq n+1$ is sufficient. Las Vergnas [21] proved a similar condition for bipartite graphs, and Yang [27] gave minimum edge density conditions to guarantee the PMH property in graphs and bipartite graphs. In the sparse setting, as a special case of a conjecture of Ruskey and Savage [26], Fink [8] proved that the hypercube has the PMH-property.

2 Sketch Proof

In this section, we sketch the proof of our main result Theorem 1.3. One of the key ingredients of the proof is a structural result that allows us to partition dense regular digraphs into (bipartite) robust expanders, which will be discussed in Section 2.1. In Section 2.2 we explain how a weaker version of Theorem 1.3 can be quickly derived from the structural result. In Sections 2.3, we briefly discuss some of the ingredients required for the full version of Theorem 1.3.

2.1 Robust Expanders

Robust expansion is a notion introduced and used by Kühn and Osthus together with several coauthors to obtain a number of breakthrough results on (di)graph decompositions and Hamiltonicity (see [20, 19, 18]). Here we present only the aspects relevant to the sketch proof and will suppress most parameters to ease exposition.

Informally, robust expanders are dense (di)graphs that are highly connected in some sense, and one of their key properties is that they are Hamiltonian under suitable (mild) degree conditions. If we could show that every d-regular digraph can be partitioned into at most $n/(qd+1)$ robust expanders where $q = 2$ if G is oriented and $q = 1$ otherwise (we use this definition of q throughout the rest of the sketch proof), it would be enough to prove Theorem 1.3. Unfortunately, it is not true, but a similar result can be obtained by generalizing a structural result of Kühn, Lo, Osthus and Staden [18] about partitioning undirected graphs into robust expanders.

In order to give the reader some sense of robust (bipartite) expansion, we give the formal definition below but note that it will not be used in the sketch proof. Also the definition we give is slightly different but equivalent to that used in other work. Let $0 < \nu \leq \tau < 1$ and suppose G is a digraph with subsets of vertices $A, B \subseteq V(G)$ (not necessarily disjoint) and $N := |A| + |B|$. We define $G[A, B]$ as the undirected bipartite graph on N vertices with bipartition A, B where, for each $a \in A$ and $b \in B$, ab is an (undirected) edge of $G[A, B]$ if and only if ab is a directed edge in $E(G)$. We say that $G[A, B]$ is a *bipartite robust* (ν, τ)-*expander* if for every $S \subseteq A$ with $\tau|A| \leq |S| \leq (1 - \tau)|A|$, the set of vertices in B having at least νN inneighbours in A (in the graph G) has size at least $|S| + \nu N$. Henceforth, we suppress the parameters ν and τ and say simply that $G[A, B]$ is a bipartite robust expander.

For any d-regular n-vertex digraph G with d linear in n, we show that it is possible to give two vertex partitions $V(G) = V_{1*} \cup \cdots \cup V_{k*}$ and $V(G) = V_{*1} \cup \cdots \cup V_{*k}$ with $k \leq 1 + n/(qd + 1)$ such that for each i, $G[V_{i*}, V_{*i}]$ is a bipartite robust expander. Letting $V_{ij} = V_{i*} \cap V_{*j}$ for all $i, j \in [k]$, note that we actually give a k^2-partition $\{V_{ij} : i, j \in [k]\}$ of $V(G)$. The following is a simplified informal version of our structural result.

Theorem 2.1. *For any $\alpha > 0$, there exists an integer $n_0 = n_(\alpha)$ such that for all d-regular digraph graphs G on $n \geq n_0$ vertices with $d \geq \alpha n$, there is a partition $\mathcal{P} = \{V_{ij} : i, j \in [k]\}$ of $V(G)$ satisfying, for all $i \in [k]$,*
(i) *$G[V_{i*}, V_{*i}]$ is a bipartite robust expander with linear minimum degree;*
(ii) *$|V_{i*}| \approx |V_{*i}|$;*
(iii) *$k \leq 1 + n/(qd + 1)$.*

2.2 A Weaker Version of Theorem 1.3

Let G be as in Theorem 1.3, i.e. an n-vertex d-regular digraph with $d \geq \alpha n$ and n sufficiently large. In this subsection we describe how Theorem 2.1 can be used to show that almost all vertices of G can be covered by at most $1 + n/(qd + 1)$ vertex-disjoint cycles (so one more cycle than stated in Theorem 1.3).

We apply Theorem 2.1 and obtain a partition $\{V_{ij} : i, j \in [k]\}$ of $V(G)$ satisfying (i)–(iii). By (ii), one can delete a small number of vertices in G so that $|V_{i*}| = |V_{*i}|$ holds for each $i \in [k]$ (for notational simplicity, we still write G and V_{ij} after deletion). A key property of bipartite robust expanders is that deleting any a small number of vertices only slightly weakens the bipartite robust expansion and minimum degree properties of (i). The following crucial observation shows that we can partition G into at most k vertex-disjoint cycles, which establishes the weaker version of Theorem 1.3 since $k \leq 1+n/(qd+1)$ by (iii).

Fix $i \in [k]$, and assume $V_{ij} = \emptyset$ for all $j \in [k] \setminus \{i\}$, i.e. $V_{i*} = V_{ii} = V_{*i}$. In this case, one can use (i) to show that $G[V_{ii}]$ is a robust expander [2] with linear minimum degree; the

[2] Roughly speaking, G is a robust expander if $G[V(G), V(G)]$ is a bipartite robust expander.

result of [20] (see also [22]) then implies that $G[V_{ii}]$ is Hamiltonian. Now, assume $V_{ij} \neq \emptyset$ for at least one $j \in [k] \setminus \{i\}$. As $|V_{i*}| = |V_{*i}|$, we have $|V_{i*} \setminus V_{ii}| = |V_{*i} \setminus V_{ii}| > 0$, so write $V_{i*} \setminus V_{ii} = \{y_1, \ldots, y_s\}$ and $V_{*i} \setminus V_{ii} = \{z_1, \ldots, z_s\}$. Let $\phi : V_{i*} \to V_{*i}$ be given by $\phi(x) = x$ for all $x \in V_{ii}$ and $\phi(y_r) = z_r$ for all $r \in [s]$. Define $G(i, \phi)$ to be the digraph whose vertices are V_{i*} and with directed edge uw present in $G(i, \phi)$ if and only if $u\phi(w)$ is present in G. In other words, $G(i, \phi)$ is the digraph obtained from $G[V_{i*}, V_{*i}]$ by identifying each $u \in V_{i*} \setminus V_{ii}$ with $\phi(u) \in V_{*i} \setminus V_{ii}$ and deleting loops. By (i), one can show that $G(i, \phi)$ is a robust expander with linear minimum degree and hence (again by [20, 22]) contains a Hamilton cycle C. Without loss of generality, assume $y_1, \ldots, y_s$ lie on C in this order. One can easily check that each path $y_r C y_{r+1}$ along C corresponds to a path in $G[V_{i*} \cup V_{*i}]$ from y_r to z_{r+1} and that these are vertex-disjoint and their union is $V_{i*} \cup V_{*i}$.

As a result, for each $i \in [k]$, we can cover $V_{i*} \cup V_{*i}$ by either a cycle (if $V_{ij} = \emptyset$ for all $j \in [k] \setminus \{i\}$) or a set of vertex-disjoint paths $\mathcal{Q}_i$ from $V_{i*} \setminus V_{ii}$ to $V_{*i} \setminus V_{ii}$. Note that the union of all these cycles and path systems gives a vertex-disjoint union of cycles covering G; indeed the path systems $\mathcal{Q}_i$ and $\mathcal{Q}_j$ only intersect in V_{ij} (where paths in $\mathcal{Q}_i$ start and paths in $\mathcal{Q}_j$ end) and in V_{ji} (where paths in $\mathcal{Q}_j$ start and paths in $\mathcal{Q}_i$ end). Our freedom to choose ϕ gives us some control over which pairs of endpoints are connected by paths in the $\mathcal{Q}_i$, and by choosing the ϕ's carefully, we can guarantee that the number of cycles in the union is at most k.

2.3 Balancing the Partition

We say $\mathcal{P} = \{V_{ij} : i, j \in [k]\}$ is a *balanced* partition of G if $|V_{i*}| = |V_{*i}|$ holds for each $i \in [k]$. Here we explain how to balance the partition $\mathcal{P}$ given in Theorem 2.1 in order that we can apply the methods described in the previous subsection. We use the idea of path contraction: consider a directed path in a digraph G and contract it so that the in- and outneighbours of the new vertex are respectively the inneighbours of the path's start vertex and the outneighbours of the path's end vertex. If the resulting graph can be partitioned into ℓ vertex-disjoint cycles, then so can G by simply uncontracting the path. Therefore, we seek a path system $\mathcal{Q}$ such that the contraction of $\mathcal{Q}$ makes the partition $\mathcal{P}$ balanced but also does not destroy the other properties of Theorem 2.1; the latter can (almost) be guaranteed by ensuring the number of edges of $\mathcal{Q}$ is small. It turns out (and is not difficult to show) that it suffices to find path systems $\mathcal{Q}_{ij}$ using edges of G from V_{i*} to V_{*j} satisfying

$$\sum_{j \neq i} e(\mathcal{Q}_{ij}) - \sum_{j \neq i} e(\mathcal{Q}_{ji}) = |V_{i*}| - |V_{*i}|.$$

We use a flow argument to find such a path system $\mathcal{Q}$.

Our argument up this point gives a collection of at most $k \leq 1 + n/(qd + 1)$ vertex-disjoint cycles that cover G, which is one more than stated in Theorem 1.3. In fact, we only get $1 + n/(qd + 1)$ cycles if Theorem 2.1 gives us a partition with $k = 1 + n/(qd + 1)$ and $V_{ij} = \emptyset$ for all $i \neq j$. By carefully making use of the additional structure in this situation, we can reduce the number of cycles by 1.

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