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On Perfect Subdivision Tilings

(Extended abstract)

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Abstract

For a given graph H, we say that a graph G has a perfect H-subdivision tiling if G contains a collection of vertex-disjoint subdivisions of H covering all vertices of G. Let $\delta_{sub}(n, H)$ be the smallest integer k such that any n-vertex graph G with minimum degree at least k has a perfect H-subdivision tiling. For every graph H, we asymptotically determined the value of $\delta_{sub}(n, H)$. More precisely, for every graph H with at least one edge, there is a constant $1 < \xi^*(H) \leq 2$ such that $\delta_{sub}(n, H) = \left(1 - \frac{1}{\xi^*(H)} + o(1)\right) n$ if H has a bipartite subdivision with two parts having different parities. Otherwise, the threshold may depend on the parity of n.

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1 Introduction

Embedding a large sparse subgraph into a dense graph is one of the most central problems in extremal graph theory. It is well-known that any graph G with minimum degree at least $\lfloor \frac{v(G)}{2} \rfloor$ has a Hamiltonian cycle, hence also a perfect matching if the number of vertices $v(G)$ of G is even. A natural generalization of a perfect matching is a perfect H-tiling, for a general graph H. We say G has a perfect H-tiling if G contains a collection of vertex-disjoint copies of H, whose union covers all vertices of G. For a positive integer n divisible by $v(H)$, we denote by $\delta(n, H)$ the minimum integer k such that any n-vertex

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graph G with minimum degree at least k has a perfect H-tiling. For any integer $r \geq 2$, the Hajnal-Szemerédi [11] theorem states that the number $\delta(n, K_r)$ is equal to $\left(1 - \frac{1}{r}\right) n$.

The minimum degree threshold of perfect tiling for general graph H was first proved by Alon and Yuster [2]. They showed that if n is divisible by $v(H)$, then $\delta(n, H) \leq \left(1 - \frac{1}{\chi(H)}\right) n + o(n)$, where $\chi(H)$ is a chromatic number of H. Komlós, Sárközy, and Szemerédi [18] improved the $o(n)$ term in Alon-Yuster theorem to some constant $C = C(H)$, which settled the conjecture of Alon and Yuster [2]. Another direction for an asymptotic extension of Hajnal-Szemerédi theorem was proved by Komlós [17]. We write the critical chromatic number of H as $\chi_{cr}(H)$, which is defined as $\chi_{cr}(H) = \frac{(\chi(H)-1)v(H)}{v(H)-\sigma(H)}$, where $\sigma(H)$ is the minimum possible size of color class in the optimal proper coloring of H. Komlós showed that for any $\gamma > 0$, there exists $n_0 = n_0(\gamma, H)$ such that if $n \geq n_0$, then for any n-vertex graph G whose minimum degree is at least $\left(1 - \frac{1}{\chi_{cr}(H)}\right) n$ contains an H-tiling which covers at least $(1 - \gamma)n$ vertices of G. Komlós [17] conjectured that the number of uncovered vertices can be reduced to a constant and this conjecture was confirmed by Shokoufandeh and Zhao [30]. More precisely, the following holds.

Theorem 1.1 (Shokoufandeh and Zhao [30])**.** *Let H be a graph. Then there exists a constant $C = C(H)$, which only depends on H such that any graph G on n-vertices with minimum degree at least $\left(1 - \frac{1}{\chi_{cr}(H)}\right) n$ contains a H-tiling which covers all but at most C vertices of G.*

The almost exact value of $\delta(n, H)$ for every graph H was determined by Kühn and Osthus [23] up to an additive constant depending only on H.

Theorem 1.2 (Kühn and Osthus [23])**.** *Let H be a graph and n be a positive integer which divisible by $v(H)$. Then there exist a constant $C = C(H)$ and $\chi(H) - 1 < \chi^*(H) \leq \chi(H)$ depending only on H such that*

$$\left(1 - \frac{1}{\chi^*(H)}\right) n - 1 \leq \delta(n, H) \leq \left(1 - \frac{1}{\chi^*(H)}\right) n + C.$$

Indeed, in [23], the authors stated how we can compute $\chi^*(H)$ for a given graph H.

1.1 Main results

Motivated by the Kühn-Osthus theorem on perfect H-tilings, several variations of Theorem 1.2 were considered. For instances, see [5, 10, 12, 13, 14, 22, 26].

We consider a problem related to the concept of perfect H-tilings and subdivision embeddings. Consider graphs G and H. We say for a graph H' is a subdivision of H if H' is obtained from H by replacing edges of H to vertex-disjoint paths. Let H and G be graphs. An *H-subdivision tiling* is a collection of disjoint union of subdivisions of H. We say that G has a *perfect H-subdivision tiling* if G has an H-subdivision tiling which covers all vertices of G. A natural question would be to determine the minimum degree threshold which ensures the existence of perfect H-subdivision tiling in any n-vertex graph G. We define this minimum degree threshold as the following.

Definition 1.3. Let H be a graph. We denote the minimum degree threshold for perfect H-subdivision tilings by $\delta_{sub}(n, H)$, which is the smallest integer k such that any n-vertex graph G with minimum degree at least k has a perfect H-subdivision tiling.

If H has no edges, then perfect H-subdivision tiling exists if and only if $v(G)$ is divisible by $v(H)$, regardless of the minimum degree $\delta(G)$ of G. Thus, from now on, we only consider graphs with at least one edge.

Since embedding bipartite graphs generally requires less minimum degree than non-bipartite graphs, we want to cover most of the vertices of the host graph with subdivisions of H that are bipartite. Suppose every bipartite subdivision of H is in some sense balanced. In that case, one cannot perfectly tile them in a highly unbalanced complete bipartite graph which has a smaller minimum degree than a balanced complete bipartite graph. For this reason, we need to measure how unbalanced bipartite subdivisions of H can be, as it poses some space barriers on the problem.

For this purpose, we introduce the following two definitions.

Definition 1.4. Let H be a graph and $X \subseteq V(H)$. We define a function $f_H : 2^{V(H)} \to \mathbb{R}$ as $f_H(X) = \frac{v(H)+e(H[X])+e(H[Y])}{|X|+e(H[Y])}$ where $Y = V(H) \setminus X$.

Definition 1.5. Let H be a graph. We define $\xi(H) \coloneqq \min\{f_H(X) : X \subseteq V(H)\}$.

Note that we always have $1 < \xi(H) \leq 2$. Another crucial factor for perfect subdivision tiling problem is the divisibility issue. Assume that all bipartite subdivisions of H have bipartitions with both parts having the same parity. If G is a complete bipartite graph $K_{a,b}$ with a, b having different parities, then we cannot find a perfect H-subdivision tiling in G, as it poses some divisibility barriers on the problem. Hence, we need to introduce the following definitions concerning the difference between two parts in bipartitions of subdivisions of H and their highest common factor.

Definition 1.6. Let H be a graph. We define $\mathcal{C}(H) \coloneqq \{(|X|+e(H[Y]))-(|Y|+e(H[X])) : X \subseteq V(H), Y = V(H) \setminus X\}$. We denote by $hcf_\xi(H)$ the highest common factor of all integers in $\mathcal{C}(H)$. (If $\mathcal{C}(H) = \{0\}$, we define $hcf_\xi(H) = \infty$.)

By considering the space and divisibility barriers, we introduce the following parameter measuring both obstacles for the problem. We will show that this is the determining factor for $\delta_{sub}(n, H)$.

Definition 1.7. Let H be a graph. We define

$$\xi^*(H) \coloneqq \begin{cases} \xi(H) & \text{if } hcf_\xi(H) = 1, \\ \max\{\frac{3}{2}, \xi(H)\} & \text{if } hcf_\xi(H) = 2, \\ 2 & \text{otherwise.} \end{cases}$$

We are now ready to state our main theorems. The following theorem gives an asymptotically exact value for $\delta_{sub}(n,H)$ except only one case, that is when $hcf_\xi(H)=2$.

Theorem 1.8. *Let H be a graph with $hcf_\xi(H)\neq 2$. For every $\gamma>0$, there exists an integer $n_0=n_0(\gamma,H)$ such that the following holds. For every integer $n\geq n_0$,*

$$\left(1-\frac{1}{\xi^*(H)}\right)n-1\leq \delta_{sub}(n,H)\leq \left(1-\frac{1}{\xi^*(H)}+\gamma\right)n$$

This theorem asymptotically determine $\delta_{sub}(n,H)$ as long as $hcf_\xi(H)\neq 2$. If $hcf_\xi(H)=2$, then the parity of n is also important. The following theorem asymptotically determines $\delta_{sub}(n,H)$ for this case.

Theorem 1.9. *Let H be a graph with $hcf_\xi(H)=2$. For every $\gamma>0$, there exists an integer $n_0=n_0(\gamma,H)$ such that the following holds. For every integer $n\geq n_0$,*

$$\begin{aligned} \frac{1}{2}n-1\leq \delta_{sub}(n,H)&\leq \left(\frac{1}{2}+\gamma\right)n && \textit{if } n \textit{ is odd,}\\ \left(1-\frac{1}{\xi^*(H)}\right)n-1\leq \delta_{sub}(n,H)&\leq \left(1-\frac{1}{\xi^*(H)}+\gamma\right)n && \textit{if } n \textit{ is even.}\end{aligned}$$

One consequence of Theorems 1.8 and 1.9 is that the value of $\delta_{sub}(n,K_r)$ behaves unpredictably when r is small. Indeed, $\delta_{sub}(n,K_2)=\left(\frac{1}{3}+o(1)\right)n$ and for each $r\in\{3,4,5\}$, we have $\delta_{sub}(n,K_r)=\left(\frac{2}{r+1}+o(1)\right)n$. For the case $r=7$, if n is even, we have $\delta_{sub}(n,K_7)=\left(\frac{1}{3}+o(1)\right)n$ otherwise, we have $\delta_{sub}(n,K_7)=\left(\frac{1}{2}+o(1)\right)n$. Finally, for every $r\geq 8$ and $r=6$, we have $\delta_{sub}(n,K_r)=\left(\frac{1}{2}+o(1)\right)n$. This is contrasting to normal H-tiling problem. This means determining factors for minimum degree thresholds of perfect H-tilings and perfect H-subdivision tilings are essentially different. Probably, the most interesting difference between the perfect H-tiling and the perfect H-subdivision tiling is that the monotonicity does not hold for subdivision tiling. For a perfect tiling, if H_2 is a spanning subgraph of H_1, then obviously $\delta(n,H_2)\leq\delta(n,H_1)$. However, for perfect subdivision tiling, this does not hold in many cases. For example, our results implies $\delta_{sub}(n,K_4)=\frac{2}{5}n+o(n)<\delta_{sub}(n,C_4)=\frac{1}{2}n+o(n)$.

As $\xi^*(H)$ is the determining factor for the minimum degree threshold, it is convenient for us to specify a bipartite subdivision achieving the value $\xi^*(H)$. We introduce the following definition.

Definition 1.10. Let H be a graph. We denote by X_H a subset of $V(H)$, where $f_H(X_H)=\xi(H)$. If there are multiple choices of X_H, we fix one choice arbitrarily. We define a graph H^* obtained from H by replacing all edges in $H[X_H]$ and $H[V(H)\setminus X_H]$ to paths of length two.

Note that H^* is a subdivision of H, which is a bipartite graph and $v(H^*)=v(H)+e(H[X_H])+e(H[V(H)\setminus X_H])$.

We observe that the inequality $\xi(H) \geq \chi^*(H^*)$ holds. Hence, if $hcf_\xi(H) \leq 2$, we may use Theorem 1.1 to find an H^*-tiling that covers all but at most constant number of vertices of G in a graph G with $\delta(G) \geq \left(1 - \frac{1}{\xi(H)} + \gamma\right) n$. In order to cover the leftover vertices, we use the absorption method. The absorption method was introduced in [28], and since then, it has been used to solve various crucial problems in extremal combinatorics. The main difficulty to apply the absorption method in our setting is that in many cases, the host graph is not sufficiently dense to guarantee that any vertices can be absorbed in the final step. To overcome this difficulty, we use the regularity lemma and an extremal result on the domination number to obtain some control over the vertices that can be absorbed.

2 Proof overview

2.1 Lower bounds

It is easy to check that the following observation holds.

Observation 2.1. *Let H be a graph and let F be a bipartite subdivision of H with bipartition (A, B). Then $\frac{|B|}{|A|} \leq \frac{1}{\xi(H)-1}$.*

This observation allows us to obtain the following proposition, which poses a lower bound for $\delta_{sub}(n, H)$ when we do not care about $hcf_\xi(H)$. Since $\xi(H)$ measures how unbalanced a bipartition of a subdivision of H can be, if the given host graph is a sufficiently unbalanced complete bipartite graph, then we cannot perfectly tile it with the subdivisions of H. Thus, we can deduce the following.

Proposition 2.2. *For every integer $n > 0$ and every graph H, there is an n-vetex graph G with minimum degree at least $\lfloor \left(1 - \frac{1}{\xi(H)}\right) n \rfloor - 1$ such that G does not have a perfect H-subdivision tiling.*

Now we cause the divisibility issue to construct a lower bound example. To obtain the lower bound in Theorem 1.8 and the first case of Theorem 1.9, we prove that the following proposition holds.

Proposition 2.3. *Let H be a graph with $hcf_\xi(H) \neq 1$. Then for every integer $n > 0$, there is an n-vertex graph G with minimum degree at least $\lfloor \frac{n}{2} \rfloor - 1$ which does not have a perfect H-subdivision tiling except for $hcf_\xi(H) = 2$ and n is even.*

The remaining case is when $hcf_\xi(H) = 2$ and n is even. The lower bound of this case can be obtained from the following.

Proposition 2.4. *For every graph H with $hcf_\xi(H) = 2$ and for every even number n, there is an n-vertex graph G with minimum degree at least $\lfloor \frac{1}{3} n \rfloor - 1$ such that G does not contain a perfect H-subdivision tiling.*

Both the proof of Propositions 2.3 and 2.4 rely on the observation that if the host graph is a complete bipartite graph with the difference between two bipartitions are not divisible by $hcf_{\xi}(H)$, then there is no perfect H-subdivision tiling. This can be verified by the definition of $hcf_{\xi}(H)$.

2.2 Upper bounds

We now sketch the proof of our main results. We first start with the following observation.

Observation 2.5. $\delta_{sub}(n, H) \leq \left(\frac{1}{2} + o(1)\right) n$.

Indeed, for every graph H, there is at least one bipartite subdivision of H. By using Erdős-Stone-Simonovits theorem and Theorem 1.2, we can deduce that $\delta_{sub}(n, H) \leq \left(\frac{1}{2} + o(1)\right) n$. As Propositions 2.2 to 2.4 provides desired lower bounds, Observation 2.5 implies that it suffices to prove the two following lemmas.

Lemma 2.6. *Let $hcf_{\xi}(H) = 1$ and n be a sufficiently large number. If $\delta(G) \geq \left(1 - \frac{1}{\xi(H)} + o(1)\right) n$, then G has a perfect H-subdivision tiling.*

Lemma 2.7. *Let $hcf(\xi)(H) = 2$ and n be a sufficiently large even number. If $\delta(G) \geq \left(\max\{\frac{1}{3}, 1 - \frac{1}{\xi(H)}\} + o(1)\right) n$, then G has a perfect H-subdivision tiling.*

In order to prove the above lemmas, we use the absorption method. Since we are dealing with a subdivision embedding problem, we define our absorber as follows.

Definition 2.8. Let H and G be graphs and take two subsets $A \subseteq V(G)$ and $X \subseteq V(G) \setminus A$. We say A is a *$Sub(H)$-absorber* for X if the both $G[A]$ and $G[A \cup X]$ have perfect H-subdivision tilings. If $X = \{v\}$, we say A is a $Sub(H)$-absorber for v.

For example, we consider an appropriate subdivision of H with an edge xy in it and add two edges vx and vy to obtain a graph H'. Then a copy of H' ensures that $V(H') - \{v\}$ is a $Sub(H)$-absorber for v. In order to establish robust absorption structures, we wish to collect many vertices that belong to many copies of such graphs H'. We will ensure this using the concept of ε-regularity.

The following is the proof outline of Lemmas 2.6 and 2.7. We omit the details of the argument as we provide them in the full version [25] of the paper.

Step 1: Preprocessing. In order to find many copies of H' containing a given vertex v, we plan to utilize the concept of ε-regularity. For this, we apply the regularity lemma and use it to obtain many disjoint ε-regular pairs covering almost all vertices. Note that those ε-regular pairs are allowed to be somewhat asymmetric. By deleting a small number of vertices, we can further ensure some minimum degree condition on every ε-regular pair. Let Z be the small set of vertices not covered by the obtained ε-regular pairs with the minimum degree condition.

Step 2: Place the absorber. In each regular pair, the ε-regularity and the minimum degree condition ensure that every vertex v in it belongs to many copies of H'. Using this property, we can find a small subset $A \subseteq (V(G) \setminus Z)$ such that A is $Sub(H)$-absorber for any small set $X \subseteq V(G) \setminus (Z \cup A)$.

Step 3: Cover almost all vertices. By considering a suitable bipartite subdivision H'' of H and applying Erdős-Stone-Simonovits theorem, we find copies of H'' disjoint from A to cover all vertices of Z as well as a small set of additional vertices. Denote the set of such vertices as W_1. As $|A \cup W_1|$ is small, the remaining graph $G \setminus (A \cup W_1)$ still has almost the same minimum degree as G. By applying Theorem 1.1, we can find $W_2 \subseteq V(G) \setminus (A \cup W_1)$ such that $G[W_2]$ has a perfect H-subdivision tiling and $|V(G) \setminus (A \cup W_1 \cup W_2)|$ is small.

Step 4: Absorb the uncovered vertices. Let $X = V(G) \setminus (A \cup W_1 \cup W_2)$. Since X is small, by our choice of A, the set A is $Sub(H)$-absorber for X. This means $G[A \cup X]$ has a perfect H-subdivision tiling. Since A, W_1, W_2 and X are vertex-disjoint sets and $A \cup W_1 \cup W_2 \cup X = V(G)$, we obtain a perfect H-subdivision tiling of G by combining $G[A \cup X]$, $G[W_1]$ and $G[W_2]$.

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