Tiling Dense Hypergraphs

(Extended abstract)

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Abstract

There are three essentially necessary conditions for perfect tilings in hypergraphs, which correspond to barriers in space, divisibility and covering. It is natural to ask when these conditions are asymptotically sufficient. Our main result confirms this for hypergraph families that are approximately closed under taking a typical induced subgraph of constant order. As an application, we recover and extend a series of well-known results for perfect tilings in hypergraphs and related settings involving vertex orderings and rainbow structures.

1 Introduction

A basic question in combinatorics is whether a combinatorial object on a ground set of vertices contains a particular substructure that spans all vertices. Since the corresponding decision problems are typically computationally intractable, the 'extremal' approach has focused on identifying easily verifiable sufficient conditions, a classic example being minimum degree conditions in the graph setting. Over the past decades, a robust literature has developed around these problems [9, 13, 21, 22], yet many questions remain widely open.

More recently, efforts have increasingly been dedicated to formulating an axiomatic approach. The idea is to identify a set of 'simple' conditions that are essentially necessary for the existence of the desired substructure. One then aims to show that satisfying these properties in a robust manner guarantees the substructure in question. Important

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milestones in this line of research are due to Keevash and Mycroft [10], Han [5] (perfect matchings) and Keevash [8] (designs) in the hypergraph setting. For graphs, analogous results have been obtained by Kühn, Osthus and Treglown [15] (Hamilton cycles), Knox and Treglown [11], Lang and Sanhueza-Matamala [16] (easily separable graphs) as well as Hurley, Joos and Lang [7] (perfect mixed tilings).

We continue this branch of research by introducing a framework for perfect tilings in hypergraphs. The literature on this subject has identified three natural barriers that prevent perfect tilings, which correspond to obstructions in space, divisibility and covering. Freschi and Treglown [4] raised the 'meta question' whether this already includes all relevant obstacles. We answer this in the affirmative for hypergraphs families (and related structures) whose features are approximately replicated by a typical subgraph of constant order. Our main contribution states that any hypergraph which robustly overcomes each of the obstructions must already contain a perfect tiling.

As an application, we recover and extend the milestones for perfect tilings under minimum degree conditions in graphs [12, 14] and hypergraphs [6, 19] as well as recent breakthroughs in the ordered setting [4], quasirandom setting [2] and the rainbow setting [1, 18].

2 A framework for hypergraph tiling

In the following, we formulate a simplified version of our main result. A *k-uniform hypergraph* (or *k-graph* for short) G consists of vertices $V(G)$ and edges $E(G)$, where each edge is a set of k vertices. Given another k-graph F, our goal is to find a *perfect F-tiling* in G, which is a collection of pairwise disjoint copies of F that together cover all vertices of G. Note that the special case, when F is a single edge, corresponds to a *perfect matching*. We denote by $\mathrm{Hom}(F; G)$ the set of *homomorphisms* from F to G, meaning the functions $\phi \colon V(F) \to V(G)$ that map edges of F to edges of G.

Obstacles for perfect tilings

Let us review three essentially necessary conditions for perfect tilings in hypergraphs.

Space. A first obstruction to perfect tilings involves space. For example, a simple instance of the *space barrier* is obtained by taking a complete graph and deleting the edges within a subset of more than half of the vertices. We formalise the corresponding space property via a linear programming relaxation. A *perfect fractional F-tiling G* is a function $\omega \colon \mathrm{Hom}(F; G) \to [0, 1]$ such that for all $v \in V(G)$, we have $\sum_{\phi \in \mathrm{Hom}(F;G)} \omega(\phi)|\phi^{-1}(v)| = 1$. Let Spa_F be the set of k-graphs with a perfect fractional F-tiling.

Divisibility. Another type of obstacle for perfect tilings arises from divisibility. For instance, it is not possible to find a perfect matching in the union of two disjoint odd cliques — a basic example of the *divisibility barrier*. We can capture this phenomenon as follows. For a homomorphism $\phi \in \mathrm{Hom}(F; G)$, denote by $\mathbf{1}_\phi \in \mathbb{N}^{V(G)}$ the indicator vector

of the image of ϕ, which satisfies $\mathbf{1}_\phi(v) = |\phi^{-1}(v)|$ for each $v \in V(G)$. Similarly, for a vertex $u \in V(G)$, denote by $\mathbf{1}_u$ the indicator vector with $\mathbf{1}_u(u) = 1$ and zero otherwise. The F-*lattice* of G is the additive subgroup $\mathcal{L}(F; G) \subseteq \mathbb{Z}^{V(G)}$ generated by the vectors $\mathbf{1}_\phi$ with $\phi \in \mathrm{Hom}(F; G)$. We say that $\mathcal{L}(F; G)$ is *complete* if it contains all *transferrals* $\mathbf{1}_v - \mathbf{1}_u$ with $u, v \in V(G)$. Denote by Div_F the set of k-graphs with complete F-lattice.

Cover. There are hypergraphs which satisfy the space and divisibility condition, but do not contain a perfect tiling simply because some vertices are not on any copy of F at all. Such a configuration is called a *cover barrier*. Motivated by this, we say that G is F-*covered* if for every vertex $v \in V(G)$, there is a homomorphism $\phi \in \mathrm{Hom}(F; G)$ such that $|\phi^{-1}(v)| = 1$. We denote by Cov_F the set of F-covered k-graphs.

Necessity. The next claim confirms that the space, divisibility and cover properties are essentially necessary for the existence of a perfect tiling. We abbreviate $m = |V(F)|$.

Observation 2.1. *If G has more than m vertices and contains a perfect F-tiling after deleting any choice of m vertices. Then G satisfies* Spa_F*,* Div_F *and* Cov_F*.*

Proof. The cover property follows trivially. The space property can be obtained by averaging over all fractional perfect F-tilings obtained after deleting m vertices. For the completeness of the lattice $\mathcal{L}(F; G)$, let $G' \subseteq G$ be obtained by deleting $m-1$ arbitrary vertices, and let $u, v \in V(G')$. By assumption, $G'-u$ has a perfect F-tiling $\mathcal{F}_u$, and $G'-v$ has a perfect F-tiling $\mathcal{F}_v$. We identify these tilings with the corresponding elements of $\mathrm{Hom}(F; G)$. It follows that $\mathcal{L}(F; G)$ contains the transferral $\mathbf{1}_v - \mathbf{1}_u = \sum_{\phi \in \mathcal{F}_v} \mathbf{1}_\phi - \sum_{\phi \in \mathcal{F}_u} \mathbf{1}_\phi$, as desired. $\square$

Sufficient conditions for perfect matchings. Now we are ready to formulate our main result, which inverts the implication of Observation 2.1. It states that every hypergraph which robustly overcomes the space, divisibility and cover barrier has a perfect tiling. Our notion of robustness is formalised with the following key definition.

Definition 2.2 (Property graph). For a k-graph G and property $\mathcal{P}$, the *property graph*, denoted by $P^{(s)}(G; \mathcal{P})$, is the s-uniform hypergraph on vertex set $V(G)$ with an edge $S \subseteq V(G)$ whenever the induced subgraph $G[S]$ satisfies $\mathcal{P}$.

Informally, we regard G as 'robustly' satisfying $\mathcal{P}$ if $P^{(s)}(G; \mathcal{P}')$ has minimum degree vertex $1 - \exp(-\Omega(s))$ where $\mathcal{P} \approx \mathcal{P}'$. However, in practise a lower degree condition suffices due the possibility of 'boosting'. Let $\delta(s)$ be the *minimum vertex degree threshold for perfect s-uniform matchings*, that is the least $\delta \in [0, 1]$ such that for all $\mu > 0$ and n large enough and divisible by s, every n-vertex s-graph P with $\delta_1(P) \geq (\delta + \mu)\binom{n-1}{s-1}$ admits a perfect matching.

Theorem 2.3. *For every k-graph F on m vertices, $s \geq 1$ and $\mu > 0$ there is n_0 such that for all $n \geq n_0$ divisible by m the following holds. Let G be a k-graph on n vertices with*

$$\delta_1\big(P^{(s)}(G; \mathrm{Spa}_F \cap \mathrm{Div}_F \cap \mathrm{Cov}_F)\big) \geq (\delta(s) + \mu)\binom{n-1}{s-1}.$$

Then G has a perfect F-tiling.

Keevash and Mycroft [10] as well as Han [5] investigated similar phenomena in the setting of perfect matchings. These results differ in their notion of robustness and in their proof techniques. In particular, Keevash and Mycroft [10] introduced the concept of completeness for lattices and used it to find a suitable allocation for the Hypergraph Blow-up Lemma. Independently, Lo and Markström [17] developed an absorption-based approach to hypergraph tiling using a (more restrictive) form of lattice completeness. Han [5] combined and extended these ideas to give a simpler proof of the Keevash–Mycroft Theorem avoiding the (Strong) Hypergraph Regularity Lemma.

Our main framework contributes to this line of research in two ways. Firstly, the interface is simple but practical. For host graph families that are approximately closed under taking typical induced subgraphs of constant order, Theorem 2.3 practically decomposes the problem of finding perfect tilings into verifying the space, divisibility and cover properties separately, which greatly simplifies the analysis. The fact that the building blocks of these properties are formulated in terms of homomorphisms adds a lot of flexibility to this approach. A more general result, which also applies to structures beyond hypergraphs is proved in the full version of the paper. In combination, we obtain short and insightful proofs of many old and new results.

The second important point about Theorem 2.3 is that the proof itself is quite short. The argument is self-contained, after discounting classic insights from combinatorics, and it does not involve the Regularity Lemma. The techniques can easily be extended to other configurations involving exceptional vertices and the partite setting. Finally, our framework can also be used to derive stability results via the theory of property testing.

Proof outline. Let us sketch the proof of Theorem 2.3. Consider a k-graph G which robustly satisfies $\mathcal{P} := \mathrm{Spa}_F \cap \mathrm{Div}_F \cap \mathrm{Cov}_F$. Our goal is to find a perfect F-tiling in G. For perfect matchings, this has been done by considering a partition $\mathcal{V}$ of $V(G)$ together with a reduced k-graph Γ on the clusters of $\mathcal{V}$, whose edges track the local edge densities of G. Under the right notion of robustness, this implies that Γ also satisfies $\mathcal{P}$. This framework allows to find a perfect matching in G either via a Hypergraph Blow-up Lemma [10] or via an absorption argument plus some classic insights on matchings in sparse graphs [5]. The main idea of our proof is to replace the reduced graph $\Gamma \in \mathcal{P}$, which approximates the whole structure of G, with a family of reduced k-graphs $\mathcal{R} \subseteq \mathcal{P}$, that describe parts of the local structure of G with higher accuracy.

To illustrate this, let us outline why G contains for some k-graph $R \in \mathcal{P}$ the blow-up R^*.[1] Recall that by assumption the property s-graph $P := P^{(s)}(G; \mathcal{P})$ is quite dense. Thus, by an old result of Erdős [3], we may find a complete s-partite s-graph $K \subseteq P$ with parts of size b where b is much larger than s. Note that each edge $S \in E(K)$ corresponds to an element $G[S]$ of $\mathcal{P}$, but for distinct edges these elements might differ or not have their vertices in the same parts of K. To deal with this, we give a colour to each of these

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[1] Meaning that R^* is obtained by replacing each vertex of R by constant number of vertices and replacing the edges with complete partite subgraphs.

configurations and apply Ramsey's theorem. This results in a subgraph $R^* \subseteq K$ with (somewhat smaller) parts of size b' and an s-vertex k-graph $R \in \mathcal{P}$ such that every edge $S \in E(R^*)$ induces an k-graph isomorphic to R with its vertices in the 'same' parts; just as desired. We informally call R^* a '$\mathcal{P}$-blow-up' with 'local reduced graph' R.

Given this observation, the proof of Theorem 2.3 proceeds in two steps implementing the Absorption Method [20]. First we match most of the vertices, then we incorporate the leftover vertices. For the first step, we show that under the assumption that the property graph has large minimum vertex degree one can partition most of the vertices of G with $\mathcal{P}$-blow-ups. We then find an almost perfect tiling in each of these blow-ups. For the second step, we show that every set of m vertices is anchored in many $\mathcal{P}$-blow-ups. This allows us to reserve a small set of $\mathcal{P}$-blow-ups beforehand to host a special structure, which can be used to absorb the leftover vertices.

The remaining challenge of the proof then consists in spelling out the embedding arguments into the blow-ups. This step is equivalent to an allocation in the context of a Blow-up Lemma applied to a 'global reduced graph' Γ. However, in our context we allocate to local reduced graph R. Since its blow-up R^* is complete partite this immediately results in the desired embedding. So in particular, we may avoid the technical details of using a (Hypergraph) Blow-up Lemma.

References

[1] Y. Cheng, J. Han, B. Wang, and G. Wang, *Rainbow spanning structures in graph and hypergraph systems*, arXiv:2105.10219 (2021).

[2] L. Ding, J. Han, S. Sun, G. Wang, and W. Zhou, *F-factors in quasi-random hypergraphs*, arXiv:2108.10731 (2021).

[3] P. Erdős, *On extremal problems of graphs and generalized graphs*, Israel Journal of Mathematics **2** (1964), 183–190.

[4] A. Freschi and A. Treglown, *Dirac-type results for tilings and coverings in ordered graphs*, Forum Math. Sigma **10** (2022), e104.

[5] J. Han, *On perfect matchings in k-complexes*, Int. Math. Res. Not. **2021** (2021), no. 11, 8741–8762.

[6] J. Han, C. Zang, and Y. Zhao, *Minimum vertex degree thresholds for tiling complete 3-partite 3-graphs*, J. Combin. Theory Ser. A **149** (2017), 115–147.

[7] E. Hurley, F. Joos, and R. Lang, *Sufficient conditions for perfect tilings*, arXiv:2201.03944 (2022).

[8] P. Keevash, *The existence of designs*, arXiv:1401.3665 (2014).

[9] ⸻, *Hypergraph matchings and designs*, arXiv:1807.05752 (2018).

[10] P. Keevash and R. Mycroft, *A geometric theory for hypergraph matching*, vol. 233, Amer. Math. Soc., 2015.

[11] F. Knox and A. Treglown, *Embedding spanning bipartite graphs of small bandwidth*, Comb. Probab. Comput. **22** (2013), no. 1, 71–96.

[12] J. Komlós, *Tiling Turán theorems*, Combinatorica **20** (2000), no. 2, 203–218.

[13] D. Kühn and D. Osthus, *Embedding large subgraphs into dense graphs*, arXiv:0901.3541 (2009).

[14] _____, *The minimum degree threshold for perfect graph packings*, Combinatorica **29** (2009), no. 1, 65–107.

[15] D. Kühn, D. Osthus, and A. Treglown, *Hamiltonian degree sequences in digraphs*, J. Combin. Theory Ser. B **100** (2010), no. 4, 367–380.

[16] R. Lang and N. Sanhueza-Matamala, *On sufficient conditions for spanning substructures in dense graphs*, to appear in Proc. Lond. Math. Soc. (2023).

[17] A. Lo and K. Markström, *F-factors in hypergraphs via absorption*, Graphs Combin. **31** (2015), no. 3, 679–712.

[18] R. Montgomery, A. Müyesser, and Y. Pehova, *Transversal factors and spanning trees*, Adv. Comb. (2022), Paper No. 3, 25.

[19] R. Mycroft, *Packing k-partite k-uniform hypergraphs*, J. Combin. Theory Ser. A **138** (2016), 60–132.

[20] V. Rödl, A. Ruciński, and E. Szemerédi, *Perfect matchings in large uniform hypergraphs with large minimum collective degree*, J. Combin. Theory Ser. A **116** (2009), no. 3, 613–636.

[21] M. Simonovits and E. Szemerédi, *Embedding Graphs into Larger Graphs: Results, Methods, and Problems*, Building Bridges II: Mathematics of László Lovász, Bolyai Society Mathematical Studies, 28. Springer, 2019, pp. 445–592.

[22] Y. Zhao, *Recent advances on Dirac-type problems for hypergraphs*, Recent trends in combinatorics, 2016, pp. 145–165.