Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

# A polynomial removal lemma for posets

(Extended abstract)

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**Abstract**

We prove a removal lemma with polynomial bound for posets. Alon and Shapira proved that every class of undirected graphs closed under the removal of edges and vertices is strongly testable. However, their bounds on the queries are not very effective, since they heavily rely on Szemerédi's regularity lemma. The case of posets turns out to be simpler: we show that every class of posets closed under the removal of edges is easily testable, that is, strongly testable with a polynomial bound on the queries. We also give a simple classification: for every class of posets closed under the removal of edges and vertices there is an $h$ such that the class is indistinguishable from the class of posets without chains of length $h$ (by testing with a constant number of queries). The analogous results hold for comparability graphs.

**DOI:** `https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-096`

## 1 Introduction

The relationship between local and global properties of structures is a central theme in combinatorics and computer science. A classical result of this kind is the triangle removal

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lemma by Ruzsa and Szemerédi [21] usually stated in the form that if a graph $G$ admits $\delta|V(G)|^3$ triangles then it can be made triangle-free by the removal of $\varepsilon|V(G)|^2$ edges, where $\delta$ depends only on $\varepsilon$. This can be applied to give a combinatorial proof of Roth's theorem [19] on 3-term arithmetic progressions, while the hypergraph removal lemma has been used to prove Szemerédi's theorem. Removal lemmas were proved for abelian groups by Green [13], for linear systems of equations by Král, Serra and Vena [16] and for permutations by Klimošová and Král [15], and by Fox and Wei [10], as well. These have led to applications in computer science allowing to test many properties by sampling a constant number of element from a structure [20].

We say that a property of (di)graphs is a set of finite (di)graphs. A (di)graph $G$ is *$\varepsilon$-far* from having a property $\Phi$ if any (di)graph $G'$ on the vertex set $V(G)$ that differs by at most $\varepsilon|V(G)|^2$ edges from $G$ does not have the property $\Phi$ either. A property $\Phi$ is *strongly testable* if for every $\varepsilon > 0$ there exists an $f(\varepsilon)$ such that if $G$ is $\varepsilon$-far from having the property $\Phi$ then for a (di)graph $G$ the induced (directed) subgraph on $f(\varepsilon)$ vertices chosen uniformly at random does not have the property $\Phi$ with probability at least one half. Alon and Shapira [3] proved that every property of undirected graphs closed under the removal of edges and vertices is strongly testable, see Lovász and Szegedy for an analytic approach [17], while Rödl and Schacht generalized this to hypergraphs [18], see also Austin and Tao [7].

Unfortunately, the dependence on $\varepsilon$ can be quite bad already in the case of undirected graphs: the known upper bounds in the Alon-Shapira theorem are wowzer functions due to the iterated involvement of Szemerédi's regularity lemma. We call a property *easily testable* if $f(\varepsilon)$ can be bounded by a polynomial of $\varepsilon$. Even triangle-free graphs are hard to test, i.e., this class is not easily testable: Behrend's construction [8] on sets of integers without 3-term arithmetic progression leads to a lower bound of magnitude $\varepsilon^{c\log(\varepsilon)}$. Alon proved that $H$-freeness is easily testable in case of undirected graphs if and only if $H$ is bipartite. For forbidden induced subgraphs Alon and Shapira gave a characterization [4], where there are very few easy cases, and ordered graphs studied by Gishboliner and Tomon are similar [11]. On the other hand, 3-colorability and, in general, "partition problems" surprisingly turned out to be easily testable, see Goldreich, Goldwasser and Ron [12]. Even a conjecture to draw the borderline between easy and hard properties seems beyond reach.

The goal of this paper is to study testability of posets as special digraphs. By a poset we mean a set equipped with a partial order that is anti-reflexive, asymmetric and transitive.

One can show that every property of posets closed under the removal of edges and vertices is strongly testable, similarly to the proof of Alon and Shapira [3], using the poset version of Szemerédi's regularity lemma proved by Hladky, Máthé, Patel and Pikhurko [14]). We show that properties of posets defined by forbidden subposets are easily testable. This is equivalent to the following removal lemma with polynomial bound. The height of a finite poset $P$ is defined as the length of its longest chain, while the width is the size of the largest antichain, denoted by $h(P)$ and $w(P)$, respectively. The chain of length $h$ is denoted by $C_h$.

Given two finite posets $P, Q$ a mapping $f : Q \to P$ is a homomorphism if it is order-preserving, i.e., $f(x) \prec f(y)$ for every $x \prec y$. The probability that a uniform random mapping from $Q$ to $P$ is a homomorphism is denoted by $t(Q,P)$.

**Theorem 1.1.** *For every $\varepsilon > 0$ and positive integers $h, w$ there exists $\delta > 0$ such that for every finite poset $Q$ of height $h$ and width $w$ and an arbitrary finite poset $P$, if $t(Q,P) < \delta$ then there exists a $Q$-free poset $P'$ on the base set of $P$ obtained by deleting at most $\varepsilon|P|^2$ edges of $P$. Moreover, $P'$ is $C_h$-free and the dependence of $\delta$ on $\varepsilon$ is polynomial.*

We use this theorem to prove testability for (not necessarily finite) classes of finite posets. The height and width of $\mathcal{P}$ of a set of finite posets are

$$h(\mathcal{P}) = \min_{P\in\mathcal{P}} h(P) \qquad w(\mathcal{P}) = \min_{\substack{P\in\mathcal{P}:\\ h(P)=h(\mathcal{P})}} w(P).$$

**Theorem 1.2.** *For every family of finite posets $\mathcal{P}$ the property of not containing any member of $\mathcal{P}$ as a subposet is easily testable. Moreover, the number of queries depends only on $h(\mathcal{P})$ and $w(\mathcal{P})$.*

We say that two properties $\Phi_1$ and $\Phi_2$ of posets are not distinguishable if for every $\varepsilon > 0$ and $i = 1, 2$ there exists $N$ such that for every poset $P$ on at least $N$ elements with property $\Phi_i$ there exists a poset $P'$ with property $\Phi_{3-i}$ such that $P'$ is obtained by deleting at most $\varepsilon|P|^2$ edges of $P$.

**Theorem 1.3.** *For every family of finite posets $\mathcal{P}$ there exists an $h$ such that the property of not containing any member of $\mathcal{P}$ as a subposet is not distinguishable from the property of not containing the chain $C_h$ as a subposet.*

Note that in our case it is meaningless to allow adding edges to the original poset, since adding edges will not change whether the poset is $\mathcal{P}$-free.

The comparability graph $G$ corresponding to a poset $P$ has vertex set $V(G) = P$ and edge set $E(G) = \{(x,y) : x \prec y \text{ or } y \prec x\}$. Alon and Fox proved that it is hard to test if a given graph is a comparability graph (or if it is perfect) [6]. Besides posets our results apply to comparability graphs, too. Given a set of (possibly infinitely many) finite graphs $\mathcal{F}$ we define the chromatic number $\chi(\mathcal{F})$ and the independence number $\alpha(\mathcal{F})$ as follows.

$$\chi(\mathcal{F}) = \min_{F\in\mathcal{F}} \chi(P) \qquad \alpha(\mathcal{F}) = \min_{\substack{F\in\mathcal{F}:\\ \chi(F)=\chi(\mathcal{F})}} \alpha(F).$$

**Theorem 1.4.** *For every family of finite graphs $\mathcal{F}$ the property of a given comparability graph not containing any member of $\mathcal{F}$ as a subgraph is easily testable. Moreover, the number of queries depends only on $\chi(\mathcal{F})$ and $\alpha(\mathcal{F})$.*

**Theorem 1.5.** *For every family of finite graphs $\mathcal{F}$ the property of being a comparability graph and not containing any member of $\mathcal{F}$ as a subgraph is not distinguishable from the property of being a comparability graph and having chromatic number at most $\chi(\mathcal{F}) - 1$.*

The proofs are based on the same ideas as in case of posets, we do not include them.

## 2 Density bounds

The complete $h$-partite poset with antichains of size $w$ will be denoted by $K_{h\times w}(= K_{w,w,\dots,w})$. In particular, $K_{h\times 1}$ is a chain of length $h$, but for this we will use the shorthand notation $C_h$.

The next lemma provides a lower bound on the density of the complete $h$-partite poset $K_{h\times w}$ in terms of the density of the chain of length $h$. The proof uses standard techniques appearing in the solution to the Zarankiewicz problem. We will use the notation $[n] := \{1, 2, \dots, n\}$.

**Lemma 2.1.** *For every poset $P$ and positive integers $h, w$ the inequality*

$$t(K_{h\times w}, P) \geq t^{w^2}(C_h, P)$$

*holds.*

*Proof.* The following two claims imply the lemma.

*Claim* 2.2.

$$t(K_{h\times w}, P) \geq t^{w}(K_{w,1,w,1,\dots}, P)$$

*Proof.* Let $(x_{i,j})_{i\in[h],j\in[w]}$ be chosen uniformly and independently at random.

$$\begin{aligned}
t(K_{h\times w}, P) &= \mathbb{P}_{(x_{i,j})_{i\in[h],j\in[w]}}\Big(\forall i' \in [h-1], j, j' \in [w] \quad x_{i',j} \prec x_{i'+1,j'}\Big)\\
&= \mathbb{E}_{\substack{(x_{i,j})_{i\in[h],j\in[w]}\\ i \text{ odd}}}\left[\mathbb{P}_{\substack{(x_{i,j})_{i\in[h],j\in[w]}\\ i \text{ even}}}\left(\forall i' \in [h-1], j, j' \in [w] \quad x_{i',j} \prec x_{i'+1,j'} \,\middle|\, (x_{i,j})_{i\in[h],j\in[w]}, i \text{ odd}\right)\right].
\end{aligned}$$

Here we split $K_{h\times w}$ into $w$ edge-disjoint copies of $K_{w,1,w,1,\dots}$. Since the events corresponding to elements in the same even layer are independent we obtain that this equals

$$\begin{aligned}
&\mathbb{E}_{\substack{(x_{i,j})_{i\in[h],j\in[w]}\\ i \text{ odd}}}\left[\mathbb{P}^{w}_{\substack{(x_{i,1})_{i\in[h]}\\ i \text{ even}}}\left(\forall i' \in [h-1], j' \in [w] \quad \substack{\text{if } i' \text{ odd then } x_{i',j'} \prec x_{i'+1,1}\\ \text{if } i' \text{ even then } x_{i',1} \prec x_{i'+1,j'}} \,\middle|\, (x_{i,j})_{i\in[h],j\in[w]}, i \text{ odd}\right)\right]\\
&\geq \mathbb{E}^{w}_{\substack{(x_{i,j})_{i\in[h],j\in[w]}\\ i \text{ odd}}}\left[\mathbb{P}_{\substack{(x_{i,1})_{i\in[h]}\\ i \text{ even}}}\left(\forall i' \in [h-1], j' \in [w] \quad \substack{\text{if } i' \text{ odd then } x_{i',j'} \prec x_{i'+1,1}\\ \text{if } i' \text{ even then } x_{i',1} \prec x_{i'+1,j'}} \,\middle|\, (x_{i,j})_{i\in[h],j\in[w]}, i \text{ odd}\right)\right]\\
&= \mathbb{P}^{w}_{\substack{(x_{i,1})_{i\in[h]} \text{ for } i \text{ even}\\ (x_{i,j})_{i\in[h],j\in[w]} \text{ for } i \text{ odd}}}\left(\forall i' \in [h-1], j' \in [w] \quad \substack{\text{if } i' \text{ odd then } x_{i',j'} \prec x_{i'+1,1}\\ \text{if } i' \text{ even then } x_{i',1} \prec x_{i'+1,j'}}\right)\\
&= t^{w}(K_{w,1,w,1,\dots}, P),
\end{aligned}$$

where we have applied Jensen's inequality. □

*Claim* 2.3.

$$t(K_{w,1,w,1,\dots}, P) \geq t^{w}(C_h, P)$$

The proof of this Claim follows the same lines as the previous one, we do not include it. The lemma follows. □

# 3 Removal lemmas with polynomial bounds

First we prove a removal lemma for chains.

**Lemma 3.1.** *For every $\varepsilon > 0$ and positive integer $h$ there exists a $\delta > 0$ such that for every finite poset $P$ if $t(C_h, P) < \delta$ then there exists a $C_h$-free poset $P'$ on the base set of $P$ obtained by deleting at most $\varepsilon|P|^2$ edges of $P$. Moreover, the dependence of $\delta$ on $\varepsilon$ is polynomial: $\delta = \left\lceil \frac{3}{\varepsilon} \right\rceil^{-h}$.*

*Remark* 3.2. For a fixed $h$ the dependence of $\delta$ on $\varepsilon$ is similar to that in the lemma. Consider a random $h$-partite poset with classes $S_1, \ldots, S_h$ of equal size large enough, where for two element $x \in S_i, y \in S_{i+1}$ with probability $h^2\varepsilon < 1$ we have $x \prec y$. The expected value of $t(C_h, P)$ is $\varepsilon^{h-1}h^{h-2}$. On the other hand, it is not hard to see that we need to remove essentially $\varepsilon|P|^2$ edges to make the poset $C_h$-free. (This is the expected number of edges between two consecutive classes.)

*Proof.* (of Lemma 3.1) Set $\gamma = \left\lceil \frac{3}{\varepsilon} \right\rceil^{-1}$. We partition the poset $P$ into classes $S_1, \ldots S_{1/\gamma}$ of size differing by at most one such that if $x \prec y$ holds for $x \in S_i$ and $y \in S_j$ then $i \leq j$. This is possible, since every finite poset has a linear extension.

Now we will delete edges in order to get a $C_h$ -free poset.

First, delete edges inside the classes – this way we delete at most $\gamma|P|^2$ edges. The remaining digraph is still a poset, denote it by $P_1$.

We define a function $r : P_1 \to [1/\gamma]$. Given an element $x \in S_i$ the integer $r(x)$ will be the largest integer such that $x$ is the maximal element of "many" chains with length $r(x)$.

Set $r(x) = 1$ for every $x \in S_1$.

Assume that for $i < 1/\gamma$ the function $r$ is defined on $\cup_{j=1}^{i} S_j$.

Given $x \in S_{i+1}$ let $r(x)$ be the largest integer such that $\left|\{y : y \prec x \text{ in } P_1, r(y) = r(x) - 1\}\right| \geq \gamma|P|$, and 1 if there is no such integer. Note that $r(x) \leq r(y)$ holds for every $x \prec y$. There are at least $(\gamma|P|)^{r(x)-1}$ chains of length $r(x)$ ending at $x$ for every $x \in S_{i+1}$ such that $r$ is strictly increasing on these chains.

Once the function $r$ is defined we delete every edge $(x, y)$ in $P_1$ for $y \prec x$ if $r(x) = r(y)$. This concerns at most $\gamma|P|^2$ edges, otherwise $r(x)$ would be larger. Note that the remaining digraph $P_2$ is still a poset and for every $x$ there are still at least $(\gamma|P|)^{r(x)-1}$ chains of length $r(x)$ ending at $x$ such that $r$ is strictly increasing on these chains.

There is no element, where $r$ takes value $(h + 1)$, since every such element would be the end of at least $(\gamma|P|)^h$ chains of length at least $(h + 1)$, but we do not have that many different chains of length $h$. By the same reason the number of elements, where $r$ takes value $h$, is at most $\gamma|P|$. We delete every edge adjacent to these elements: this way we delete at most $\gamma|P|^2$ edges, denote the remaining poset by $P'$.

The total number of edges deleted is at most $3\gamma|P|^2 < \varepsilon|P|^2$.

The poset $P'$ does not contain any chain of length at least $h$, since edges where the value of $r$ at the end-vertex is at least $h$ has been deleted, while edges where the value of

$r$ at the end-vertex is not greater than at the starting vertex have also been deleted. This finishes the proof of the lemma. $\square$

*Proof.* (of Theorem 1.1) Set $\delta = \lceil \frac{3}{\varepsilon} \rceil^{-hw^2}$. The poset $Q$ is a subposet of $K_{h\times w}$, hence Lemma 2.1 implies $\delta > t(Q, P) \geq t(K_{h\times w}, P) \geq t^{w^2}(C_h, P)$. By Lemma 3.1 there exists a $C_h$-free subposet $P'$ of $P$ obtained by deleting at most $\varepsilon|P|^2$ edges. $\square$

**Corollary 3.3.** *For every $\varepsilon > 0$ and positive integers $h, w$ there exists $\delta > 0$ such that for every finite graph $F$ of chromatic number $h$ and independence number $w$ and an arbitrary finite comparability graph $G$ if $t(F, G) < \delta$ then there exists an $F$-free comparability graph $G'$ on the vertex set of $G$ obtained by deleting at most $\varepsilon|V(G)|^2$ edges of $G$. Moreover, $G'$ is $K_h$-free and the dependence on $\varepsilon$ is polynomial: $\delta = \left\lceil \frac{3}{\varepsilon} \right\rceil^{-hw^2}$.*

*Proof.* The graph $F$ is a subgraph of the multipartite Turán graph $T$ with $h$ classes each of size $w$, hence $t(F, G) \geq t(T, G)$. Let $P$ be one of the posets whose comparability graph is $G$. Note that $t(T, G) \geq t(K_{h\times w}, P)$, since we may assume that $T$ is the comparability graph of $K_{h\times w}$, hence every homomorphism of $K_{h\times w}$ to $P$ is a comparability-preserving map from $T$ to $G$, i.e., a graph homomorphism.

We obtain by Lemma 2.1 that $\delta > t(F, G) \geq t(K_{h\times w}, P) \geq t^{w^2}(C_h, P)$.

Lemma 3.1 implies that there exists a $C_h$-free subposet $P'$ of $P$ obtained by deleting at most $\varepsilon|P|^2$ edges, and its comparability graph $G'$ satisfies the conditions of the corollary. $\square$

# 4 Property testing

*Proof.* (of Theorem 1.2) Set $h = h(\mathcal{P})$ and $w = w(\mathcal{P})$. Consider an $\varepsilon > 0$ and a poset $P$ such that after removing $\varepsilon|P|^2$ the resulting poset still contains a subposet in $\mathcal{P}$. By Corollary 3.3 the probability that $hw$ elements chosen uniformly at random contain $K_{h\times w}$, and hence a poset in $\mathcal{P}$ as a subposet is at least $\left\lceil \frac{3}{\varepsilon} \right\rceil^{-hw^2}$. If we choose $hw \left\lceil \frac{3}{\varepsilon} \right\rceil^{hw^2}$ elements uniformly at random then the probability of finding a poset in $\mathcal{P}$ as subposet is more than a half. $\square$

*Proof.* (of Theorem 1.3) If a poset does not contain the chain $C_{h(\mathcal{P})}$ as a subposet then it does not contain any poset from $\mathcal{P}$.

In order to prove the other direction consider a poset $Q \in \mathcal{P}$ with height $h(\mathcal{P})$. If a poset $P$ does not contain $Q$ as a subposet then there is no injective homomorphism from $Q$ to $P$, hence $t(Q, P) \leq |P|^{-1}|Q|^2$. Theorem 1.1 shows that by the removal of $3|P|^{-1/(h(Q)w(Q)^2)}|Q|^{1/(2h(Q)w(Q)^2)}|P|^2$ edges from $P$ one obtains a $C_{h(\mathcal{P})}$-free poset. $\square$

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