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Graph Covers and Generalized Snarks

(Extended abstract)

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Abstract

The notion of graph cover, also known as locally bijective homomorphism, is a discretization of covering spaces known from general topology. It is a pair of incidence-preserving vertex- and edge-mappings between two graphs, the edge-component being bijective on the edge-neighborhoods of every vertex and its image. In line with the current trends in topological graph theory and its applications in mathematical physics, graphs are considered in the most relaxed form and as such they may contain multiple edges, loops and semi-edges.

Nevertheless, simple graphs (binary structures without multiple edges, loops, or semi-edges) play an important role. The Strong Dichotomy Conjecture of Bok et al. [2022] states that for every fixed graph H, deciding if an input graph covers H is either polynomial time solvable for arbitrary input graphs, or NP-complete for simple ones. These authors introduced the following quasi-order on the class of connected graphs: A connected graph A is called *stronger than* a connected graph B if every simple graph that covers A also covers B. Witnesses of A not being stronger than B are *generalized snarks* in the sense that they are simple graphs that cover A but do not cover B. Bok et al. conjectured that if A has no semi-edges, then A is stronger than B if and only if A covers B. We prove this conjecture for cubic one-vertex graphs, and we also justify it for all cubic graphs A with at most 4 vertices.

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1 Introduction

Combinatorial treatment of graph coverings had its primary incentive in the solution of Heawood's Map Colour Problem due to Ringel, Youngs and others [18, 17]. That coverings underlie the techniques that led to the eventual solution of the problem was recognized by Alpert and Gross [9]. These ideas further crystallized in 1974 in the work of Gross [8] where voltage graphs were introduced as a means of a purely combinatorial description of regular graph coverings. In parallel, motivated by the effort to construct infinite families of highly symmetrical graphs, Biggs came with the very same idea which appeared in monograph [2]. Much of the theory of combinatorial graph coverings in its own right was subsequently developed by Gross and Tucker in the 1970's. We refer the reader to [10, 20] and the references therein.

In [13] the combinatorial theory of graph coverings and voltage assignments was established and extended onto a more general class of graphs which include edges with free ends (called semi-edges). The new concept of a graph proved to be useful in applications as well as in theoretical considerations.

Nowadays, the construction of a graph covering over a prescribed graph is established as a useful technique allowing to construct effectively infinite families of graphs sharing prescribed properties. In particular, it was used to construct extremal regular graphs with fixed degree and diameter [15], to construct cages and their approximations [11], and in investigation of flows on graphs [16] .

From the computational complexity point of view, Bodlaender [3] showed that deciding if a given graph G covers a given graph H (both graphs are part of the input) is NP-complete. Abello et al. [1] asked for the complexity of this question when parameterized by the target graph H. They gave the first examples of graphs for which the problem, referred to as H-Cover, is NP-complete or polynomial time solvable. It should be noted that in this seminal paper, both the parameter and the input graphs are allowed to have loops and multiple edges, but not semi-edges. The impact of semi-edges for the complexity issues is first discussed in Bok et al. [5]. It is perhaps somewhat surprising that in all cases where the complexity of the H-Cover problem is known to be NP-complete, it has been proved NP-complete even for simple graphs on input. This has been now conjectured to hold true in general, as the Strong Dichotomy Conjecture (cf. Conjecture 1 below) in [6].

Bok et al. [4] have proved that the Strong Dichotomy Conjecture holds true for all graphs, provided it holds true for connected ones. The curiosity of the NP-hardness reduction is its non-constructiveness. For two graphs A and B, they use a simple graph A' which covers A and does not cover B, if such a simple graph exists (and an arbitrary simple cover of A otherwise). However, there is no clue how to decide if such an A' exists or not. (This paradox is not undermining the reduction, because the approach is used for fixed graphs A and B, two of the connected components of the target graph H, to prove the existence of a polynomial time reduction between computational problems.) As a consequence, they defined a binary relation between connected graphs, saying that a graph A is stronger than a graph B if such a graph A' does not exist. Our aim is to contribute to the study of this relation, which shows a surprising connection to the well studied area of edge-colorings of graphs. In order to describe this connection, let us remind the reader of the (graph-theoretical) notion of a *snark*.

A snark is a simple 2-connected cubic graph which is not 3-edge-colorable. The interest in snarks was boosted by an observation by Heawood (1890) that the Four Color Theorem is equivalent to the statement: there are no planar snarks. For about 80 years only few examples of non-trivial snarks were constructed until Isaacs (1975) introduced the infinite family of flower snarks and the operation of dot-product allowing to construct a new nontrivial snark from two given ones. Investigation of snarks is an active area of research due to the fact that many long-standing conjectures on graphs (such as the 5-flow conjecture or the cycle double cover conjecture) can be reduced to problems on snarks, see [19, 7, 14] and the references therein.

If we denote by $F(3,0)$ the one-vertex graph with 3 semi-edges, and by $F(1,1)$ the one-vertex graph with 1 semi-edge and 1 loop, a snark is a simple cubic graph that covers $F(1,1)$ but does not cover $F(3,0)$, i.e., a witness for the fact that $B = F(1,1)$ is not stronger than $A = F(3,0)$. By the Petersen theorem, a 2-connected cubic graph always contains a perfect matching and hence it covers $F(1,1)$. In this sense every witness A' for A not being stronger than B can be viewed as a *generalized snark*.

It is easy to see that A is stronger than B whenever A covers B. For all known pairs A, B such that A is stronger than B and A does not cover B, the graph A contains semi-edges. In [12] it is conjectured that this is always the case (cf. Conjecture 2 below). In this paper, we justify these conjectures in several general situations, namely for cubic graphs.

2 Preliminaries

Definition 1. *A* graph *is a finite set of vertices accompanied with a finite set of* edges*, where an edge is either a* loop*, or a* semi-edge*, or a* normal edge*. A normal edge is incident with two distinct vertices and adds 1 to the degree of each of them. A loop is incident with a single vertex and adds 2 to its degree. A semi-edge is also incident with a single vertex, but adds only 1 to its degree.*

As defined, we only consider undirected graphs. However, graphs may have multiple loops and/or multiple semi-edges incident with the same vertex, and also multiple normal edges incident with the same pair of vertices. A graph is called *simple* if it has no loops, no semi-edges and no multiple normal edges. The *edge-neighborhood* $E_G(u)$ of a vertex u is the set of edges of G incident with u.

Definition 2. *A* covering projection *from a graph G to a connected graph H is a pair of surjective mappings $f_V\colon V(G) \to V(H)$ and $f_E\colon E(G) \to E(H)$ such that*

- f_E maps semi-edges onto semi-edges and loops onto loops, respectively, (normal edges may be mapped onto normal edges, loops, and semi-edges),

- f_E is incidence preserving (i.e., if $e \in E(G)$ is incident with vertices $u, v \in V(G)$, then $f_E(e)$ is incident with $f_V(u)$ and $f_V(v)$, which may of course be the same vertex),

- f_E is a local bijection on the edge-neighborhoods of any vertex and its image.

The last condition implies that f_V is degree preserving and, together with the other conditions, that the preimage of a normal edge incident with vertices $u, v \in V(H)$ is a disjoint union of normal edges, each incident with one vertex in $f_V^{-1}(u)$ and with one vertex in $f_V^{-1}(v)$, spanning $f_V^{-1}(u) \cup f_V^{-1}(v)$; the preimage of a loop incident with vertex $u \in V(H)$ is a disjoint union of cycles spanning $f_V^{-1}(u)$ (a loop is a cycle of length 1, and two parallel edges form a cycle of length 2); and the preimage of a semi-edge incident with a vertex $u \in V(H)$ is a disjoint union of semi-edges and normal edges spanning $f_V^{-1}(u)$.

If a graph G allows a covering projection onto a graph H, we say that G *covers* H, and we write $G \to H$.

It is well known that in a covering projection to a connected graph, the sizes of preimages of all vertices of the target graph are the same. This implies that $|V(H)|$ divides $|V(G)|$ whenever $G \to H$ for a connected graph H. We say that G is a k-fold cover of H, with $k = \frac{|V(G)|}{|V(H)|}$ in such a case. It follows that both vertex- and edge-component of a covering projection into a connected graph are surjective mappings.

We are interested in the following computational problem, parameterized by the target graph H.

PROBLEM:	H-COVER
INPUT:	A graph G.
QUESTION:	Does G cover H?

Abello et al. [1] raised the question of characterizing the complexity of H-COVER for simple graphs H. Despite intensive effort and several general results, the complete characterization and even a conjecture on what are the easy and hard cases is not in sight. Bok et al. [5] was the first paper that studied this question for (multi)graphs with semi-edges. A polynomial/NP-completeness dichotomy is believed in, and it has been conjectured in a stronger form in [6]:

Conjecture 1 (Strong Dichotomy Conjecture). For every graph H, the H-COVER problem is either polynomial-time solvable for general input graphs, or NP-complete for simple graphs on input.

In this connection, the following relation among graphs introduced in [4] seems to play quite an important role.

Definition 3. *A connected graph A is said to be* stronger than *a connected graph B, denoted by $A \triangleright B$, if it holds true that any simple graph covers A only if it also covers B. Formally,*

$$A \triangleright B \quad \Leftrightarrow \quad \forall G \textit{ simple graph} : ((G \to A) \Rightarrow (G \to B)).$$

It follows from the definition (and from the fact that the composition of covering projections is also a covering projection) that $A \triangleright B$ whenever $A \to B$. Moreover, if A is a simple graph, then $A \triangleright B$ if and only if $A \to B$. The graphs $F(3,0)$ and $F(1,1)$ defined in the Introduction provide an example of graphs such that $F(3,0) \triangleright F(1,1)$ though $F(3,0) \not\to F(1,1)$.

It would certainly be too ambitious a goal to try to understand the complexity of the "being stronger" relation, as understanding the complexity of the $\triangleright$ relation would require a full understanding of covering graphs by simple graphs, which is known to be NP-complete for many instances of the target graphs. However, there may be a hope for understanding $A \triangleright B$ for those pairs of graphs A, B such that $A \not\to B$. In the problem session of GROW 2022 workshop in Koper, September 2022, we have conjectured that the presence of semi-edges in A is vital in this sense (cf. [12]).

Conjecture 2. If A has no semi-edges, then $A \triangleright B$ if and only if $A \to B$.

3 Our results

The goal of this paper is to justify the above mentioned conjectures for several general situations. We first show that A cannot be much smaller than B in order to be stronger than it. Then we prove the conjectures for bipartite two-vertex graphs A.

Theorem 1. *Let A and B be connected graphs such that $A \triangleright B$. Then $|V(B)|$ divides $2|V(A)|$. If, moreover, A has no semi-edges, then $|V(B)|$ divides $|V(A)|$.*

Theorem 2. *Let A be a dipole, i.e., a graph with two vertices joined by d parallel edges. Then for every graph B, $A \triangleright B$ implies $A \to B$.*

We further pay a closer attention to cubic graphs. By a technical case analysis, which involves construction of several generalized snarks, we prove that Conjectures 2 and 3 hold true for all graphs A with at most 4 vertices (and all graphs B). Our last two results prove the conjectures for all one-vertex cubic graphs B (i.e., for $B = F(3,0)$ and $B = F(1,1)$) and arbitrary A by actually completely describing the graphs A (even those with semi-edges) such that $A \triangleright B$.

Theorem 3. *For any connected graph A, it holds true that $A \triangleright F(3,0)$ if and only if $A \to F(3,0)$.*

For the last result, we need to introduce a new notion. A *semi-covering projection* from a graph G to a graph H is a pair of vertex and edge mappings which are incidence preserving (like covering projections), but semi-edges are allowed to be mapped on loops and the preimage of a loop in H is allowed to be any 2-regular subgraph of G spanning the preimage of the vertex incident with the loop in H (i.e., a disjoint union of cycles, digons, loops and open paths). We write $G \rightsquigarrow H$ when G allows a semi-covering projection onto H. With the help of this notion we can describe the graphs that are stronger than $F(1,1)$. (Note that a graph without semi-edges which semi-covers $F(1,1)$ also covers $F(1,1)$.)

Theorem 4. *For any connected graph A, it holds true that $A \triangleright F(1,1)$ if and only if $A \rightsquigarrow F(1,1)$.*

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