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Trees of Intermediate Volume Growth

(Extended abstract)

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Abstract

For every sufficiently well-behaved function $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that grows at least linearly and at most exponentially we construct a tree T of uniform volume growth g, that is,

$$C_1 \cdot g(r/4) \leq |B_G(v,r)| \leq C_2 \cdot g(4r), \quad \text{for all } r \geq 0 \text{ and } v \in V(T),$$

with $C_1, C_2 > 0$ and where $B_G(v,r)$ denotes the ball of radius r centered at a vertex v. In particular, this yields examples of trees of uniform intermediate (*i.e.,* super-polynomial and sub-exponential) volume growth.

We use this construction to provide first examples of unimodular random rooted trees of uniform intermediate growth, answering a question by Itai Benjamini. We find a peculiar change in structural properties for these trees at growth $r^{\log\log r}$.

Our results can be applied to obtain triangulations of $\mathbb{R}^2$ with varied growth behaviours and a Riemannian metric on $\mathbb{R}^2$ for the same wide range of growth behaviors.

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1 Introduction

For a graph G, a vertex $v \in V(G)$ and $r \geq 0$, the set $B_G(v,r) := \{w \in V(G) \mid d_G(v,w) \leq r\}$ is the *ball* of radius r around v. The growth of these balls as r increases is the *growth behavior* or *volume growth* of G at the vertex v. The two extreme cases of growth are the regular trees (of exponential growth) and the lattice graphs (of polynomial growth).

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It is an ongoing endeavor to map the growth behaviors in various graph classes, the most famous example being Cayley graphs of finitely generated groups (see *e.g.* [8]). Major results in this regard are the existence of Cayley graphs of *intermediate* growth (that is, super-polynomial but sub-exponential) [7], and the proof that vertex-transitive graphs only have polynomial growth for integer exponents [11, Theorem 2].

Vertex-transitive graphs have the same growth at every vertex. In other graph classes this must be imposed more explicitly: following [5], a graph G is of *uniform* growth $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ if there are constants $c_1, C_1, c_2, C_2 \in \mathbb{R}_{>0}$ so that

$$C_1 \cdot g(c_1 r) \;\leq\; |B_G(v,r)| \;\leq\; C_2 \cdot g(c_2 r) \quad \text{for all } r \geq 0 \text{ and } v \in V(G). \tag{1.1}$$

In this article we construct trees for a wide range of growth behaviors, including intermediate and polynomial with non-integer exponents. The question of uniform intermediate growth for trees was initially posed by Itai Benjamini (private communication) in the context of unimodular random trees. However, even the existence of deterministic trees of such growth was unknown at that time.

We firstly verify the existence in the deterministic case for various growth behaviors. We then demonstrate that our construction extends to unimodular random rooted trees with the same wide range of growth behaviors, answering Benjamini's question in the positive. Finally, we probe the structure of these unimodular trees and find a threshold phenomenon happening at growth roughly $r^{\log\log r}$. As an application, in Section 4.1 we obtain triangulations of the plane, as well as Riemannian metrics on $\mathbb{R}^2$, both with the same wide range of growth behaviors.

Our work follows a history of studies on the growth rate of graphs, and particularly of trees. The first (unimodular) trees of uniform polynomial growth were constructed by Benjamini and Schramm [6]. Special attention to exponential growth for trees was given by Timár [10], focusing on the existence of a well-defined exponential rate. Recent advancement in this regard was made by Abert, Fraczyk and Hayes [1]. Intermediate but not necessarily uniform growth in trees has been studied by Amir and Yang [3] as well as the references given therein.

1.1 Motivation

The interest in such trees originates in the observation, made by physicists, that planar triangulations can have non-quadratic uniform growth [2, 4]. In their landmark paper [6] Benjamini and Schramm explained this curious phenomenon by constructing trees of every polynomial growth and then demonstrating how any tree of a particular growth can be turned into such a triangulation with a similar growth:

Construction 1.1. Suppose T is a tree of maximum degree $\bar{\Delta}$. Fix a triangulated sphere with at least $\bar{\Delta}$ pairwise disjoint triangles. Take copies of this sphere, one for each vertex of T, and identify two spheres along a triangle when the associated vertices are adjacent in T. This yields a planar triangulation. If T is of uniform growth g, so is this triangulation.

As we explain at the concluding remarks, our trees can be similarly adapted to produce triangulations of the plane.

1.2 Main results

We show the existence of deterministic and unimodular random rooted trees with growth $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for various functions between polynomial and exponential growth:

Theorem 1. *If $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is super-additive and (eventually) log-concave, then there exists a deterministic tree T of uniform growth g.*

Theorem 2. *If $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is super-additive and (eventually) log-concave, then there exists a unimodular random rooted tree $(\mathcal{T}, \omega)$ of uniform growth g.*

Super-additivity and log-concavity formalize the constraints on prescribed growth to be "at least linear" and "at most exponential" and prevent certain pathologies, such as unbounded degree or too strong oscillations in the growth behavior. Concretely, a function g is *super-additive* if $g(x+y) \geq g(x)+g(y)$ for every x, y in its domain, and it is *log-concave* if $g(tx + (1-t)y) \geq g(x)^t g(y)^{1-t}$ for every x, y in its domain and every $t \in (0, 1)$.

We also prove a structure theorem (Theorem 3) that describes the structure of our unimodular trees depending on the prescribed growth rate. We show that the growth rate $r^{\log \log r}$ acts as a threshold, with "faster-growing" trees being *apocentric* (not unlike the classical canopy tree) and "slower-growing" trees being *balanced* (with a precise definition of these terms in Construction 2.3). In both cases, the trees are a.s. 1-ended for most prescribed growths.

2 The construction

For each integer sequence $\delta_1, \delta_2, \delta_3, ... \in \mathbb{N}$ with $\delta_n \geq 1$ we construct a tree $T = T(\delta_1, \delta_2, ...)$. The choice of sequence will determine the growth rate of T.

Construction 2.1. The trees T_n are defined recursively. In each tree we distinguish two special types of vertices: a *center*, and the *apocentric vertices*. Both will be defined alongside the trees:

(*i*) T_0 is a single vertex, which is the center of T_0 and an apocentric vertex.

(*ii*) T_n is built from $\delta_n + 1$ disjoint copies $\tau_0, \tau_1, ..., \tau_{\delta_n}$ of T_{n-1} that we join into a tree by adding the following edges: for each $i \in \{1, ..., \delta_n\}$ add an edge between the center of τ_i and some apocentric vertex of τ_0.

There is a choice in selecting these apocentric vertices of τ_0 (and we can choose the same apocentric vertex more than once), but we require that these adjacencies be distributed in a uniform way among the apocentric vertices of τ_0.

The center of T_n is that of τ_0; the apocentric vertices of T_n are those of $\tau_1, ..., \tau_{\delta_n}$.

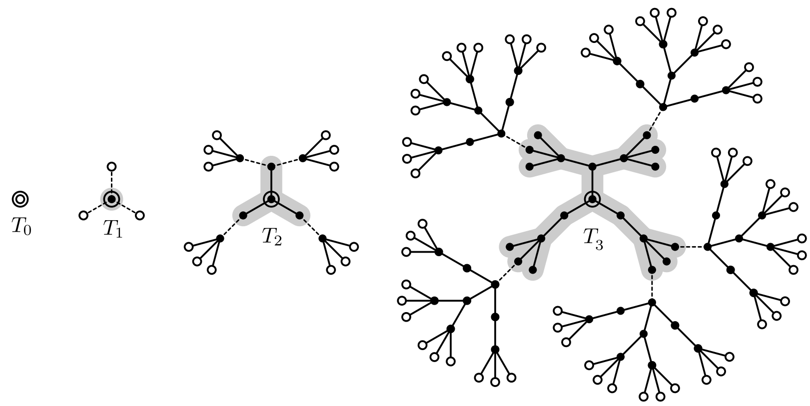

Figure 1: The first four trees $T_0, ..., T_3$ defined by $\delta_n := n + 2$. The ringed vertex is the center, and the white vertices are the apocentric vertices in the respective tree. The highlighted subgraph is the central copy τ_0 in T_n. The dashed lines are the new edges connecting the copies to form a single tree.

Observation 2.2.

(i) T_n has exactly $(\delta_1 + 1) \cdots (\delta_n + 1)$ vertices;

(ii) T_n has exactly $\delta_1 \cdots \delta_n$ apocentric vertices, all of which are leaves of the tree;

(iii) the distance from the center of T_n to any of its apocentric vertices is $2^n - 1$.

Construction 2.3. For each $n \geq 1$ identify T_n with one of its copies $\tau_0, \tau_1, ..., \tau_{\delta_{n+1}}$ in T_{n+1}. In this way we obtain an inclusion chain $T_0 \subset T_1 \subset T_2 \subset \cdots$ and the union $T = T(\delta_1, \delta_2, ...) := \bigcup_{n \geq 0} T_n$ is an infinite tree.

For later use we distinguish three natural types of limits:

- the *centric limit* always identifies T_n with the "central copy" τ_0 in T_{n+1}. This limit comes with a designated vertex $x^* \in V(T_0) \subset V(T)$, the *global center*.
- *apocentric limits* always identify T_n with an "apocentric copy" τ_i in T_{n+1}.
- *balanced limits* make infinitely many central and apocentric identifications.

We show that for a suitable sequence $\delta_1, \delta_2, \delta_3, ... \in \mathbb{N}$ and independently of the type of the limit, the tree T has a uniform volume growth, and that with a deliberate choice of the sequence we can model a wide range of growth behaviors, including polynomial, intermediate and exponential. This proves Theorem 1.

The following example computation gives an idea of the connection between the sequence $\delta_1, \delta_2, \delta_3, ... \in \mathbb{N}$ and the growth of T. Let T be the centric limit with global center $x^* \in V(T)$. By Observation 2.2 (iii) the ball of radius $r = 2^n - 1$ in T, centered at x^*, is exactly $T_n \subset T$. By Observation 2.2 (i) it follows that

$$|B_T(x^*, r)| = |T_n| = (\delta_1 + 1) \cdots (\delta_n + 1). \tag{2.1}$$

So, if we aim for $B_T(x^*, r) \approx g(r)$ with a given growth function $g\colon \mathbb{R}_0 \to \mathbb{R}_0$, then (2.1) suggests to use a sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$ for which $(\delta_1 + 1) \cdots (\delta_n + 1)$ approximates $g(2^n - 1)$. It turns out to be more convenient to approximate $g(2^n)$, so

$$\delta_n + 1 \approx \frac{g(2^n)}{g(2^{n-1})}, \tag{2.2}$$

where we introduce an error when rounding the right side to an integer.

To establish uniform growth with a prescribed growth rate g it remains to prove:

- the error introduced by rounding the right side of (2.2) is manageable;
- an estimation close to (2.1) holds for radii r that are not of the form $2^n - 1$;
- an estimation close to (2.1) holds for general limit trees and around vertices other than a designated "global center".

We address each of these points in our paper for g super-additive and log-concave.

We close with three examples demonstrating the versatility of Construction 2.3.

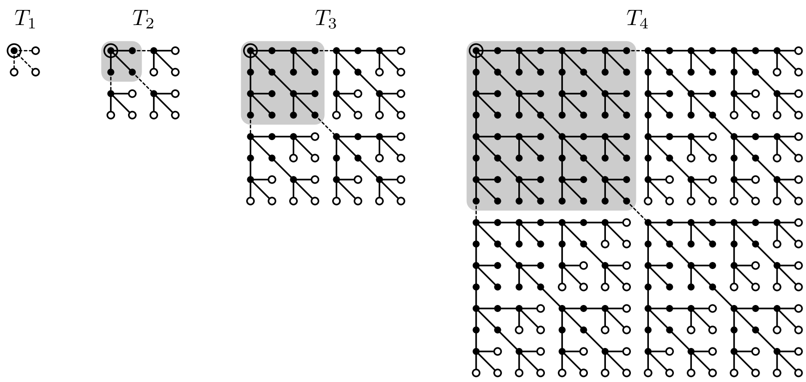

Figure 2: $T(3, 3, \ldots)$ embedded in the square of the 2D lattice.

Example 2.4 (Polynomial growth)**.** If we aim for polynomial growth $g(r) = r^\alpha, \alpha \in \mathbb{N}$ then the heuristics (2.2) suggests to use a constant sequence $\delta_n := 2^\alpha - 1$.

The corresponding trees T_n embed in the α-th powerof the α-dimensional lattice.

More generally, for any constant sequence $\delta_n := c$ we expect to find polynomial volume growth, potentially with a non-integer exponent $\log(c + 1)$.

Example 2.5 (Exponential growth)**.** For $\delta_n := d^{2^{n-1}}, d \in \mathbb{N}$ the centric limit T is the d-ary tree,

of exponential volume growth. Using (2.1) for $r = 2^n$ we find

$$|B_T(x^*, r - 1)| = (\delta_1 + 1) \cdots (\delta_n + 1) = \prod_{k=1}^{n} \left(d^{2^{k-1}} + 1 \right) = \sum_{i=0}^{2^n - 1} d^i = \frac{d^{2^n} - 1}{d - 1} = \frac{d^r - 1}{d - 1}.$$

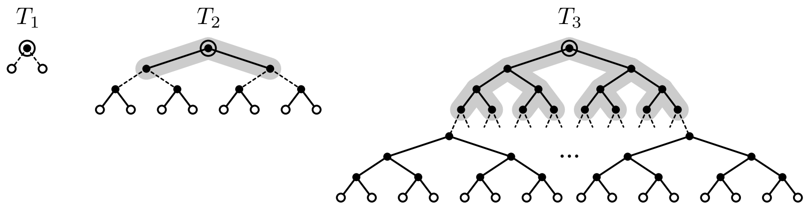

Figure 3: The binary tree constructed from Construction 2.1 using the sequence $\delta_n = 2^{2^{n-1}}$.

Extrapolating from Example 2.4 and Example 2.5, it seems reasonable that unbounded sequences $\delta_1, \delta_2, \delta_3, \ldots$ with a growth sufficiently below doubly exponential result in intermediate volume growth.

Example 2.6 (Intermediate growth)**.** For $\delta_n := (n+3)^\alpha - 1$, $\alpha \in \mathbb{N}$ we can compute this explicitly (see Figure 1 for the case $\alpha = 1$). If T is the centric limit with global center $x^* \in V(T)$ and $r = 2^n$, then:

$$\begin{aligned}|B_T(x^*, r-1)| = (\delta_1 + 1)\cdots(\delta_n + 1) = (\tfrac{1}{6}(n+3)!)^\alpha \sim (n!n^3)^\alpha \sim (n^n e^{-n} n^{7/2})^\alpha \\ = r^{\alpha \log\log r} r^{-\alpha/\ln 2} (\log r)^{7\alpha/2}\end{aligned}$$

Indeed, by Theorem 1, this choice of sequence leads to a tree of uniform intermediate volume growth. Trees constructed from $\delta_n \sim n^\alpha$ present an interesting boundary case in Section 3 when we discuss unimodular random trees (see also Theorem 3).

3 Passing to unimodular random trees

A *rooted graph* is a pair of the form (G, o), where G is a graph and $o \in V(G)$ is "a root". For a definition of *random rooted graphs* we follow [6]: firstly, there is a natural topology on the set of rooted graphs – the *local topology* – induced by the metric

$$\operatorname{dist}\big((G,o),(G',o')\big) := 2^{-R} \quad \text{if } B_G(o,r) \cong B_{G'}(o',r) \text{ for all } 0 \le r \le R \text{ and } B_G(o,R+1) \ncong B_{G'}(o',R+1),$$

where it is understood that $B_G(o,r)$ is rooted at o and that isomorphisms between rooted graphs preserve roots.

A random rooted graph (G, o) is a Borel probability measure (for the local topology) on the set of locally finite, connected rooted graphs. We call (G, o) *finite* if the set of infinite rooted graphs has (G, o)-measure zero. If in addition the conditional distribution of the root in (G, o) over each finite graph is uniform, then (G, o) is called *unbiased*.

Given a sequence (G_n, o_n) of unbiased random rooted graphs, a random rooted graph (G, o) is said to be the *Benjamini-Schramm limit* of (G_n, o_n) if for every finite rooted graph (H, ω) and natural number $r \ge 0$ we have

$$\lim_{n\to\infty} P\big(B_{G_n}(o_n, r) \cong (H,\omega)\big) = P\big(B_G(o,r) \cong (H,\omega)\big).$$

If it exists, (G, o) is the unique limit. If a random rooted graph is the Benjamini-Schramm limit of some sequence, we say that it is *sofic*.

One can show that a set of graphs of uniformly bounded degree is compact in the local topology, and thus, a sequence (G_n, o_n) of uniformly bounded degree always has a convergent subsequence.

We say that a random rooted graph (G, o) is of uniform growth $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, if there are constants $c_1, C_1, c_2, C_2 \in \mathbb{R}_{>0}$ so that a.s.

$$C_1 \cdot g(c_1 r) \leq |B_G(o, r)| \leq C_2 \cdot g(c_2 r), \quad \text{for all } r \geq 0.$$

A random rooted graph (G, o) is *unimodular* if it obeys the *mass transport principle*, *i.e.*,

$$\mathbb{E}\Big[\sum_{x \in V(G)} f(G, o, x)\Big] = \mathbb{E}\Big[\sum_{x \in V(G)} f(G, x, o)\Big]$$

for every *transport function* f, which, for our purpose, are sufficiently defined as Borel functions over doubly-pointed graphs that output non-negative real numbers (for a precise definition we direct the reader to [9]).

The function f simulates mass transport between vertices, and the mass transport principle states, roughly, that the root o sends, on average, as much mass to other vertices as it receives from them. Unimodular graphs are significant in the theory of random graphs and encompass some important classes, notably, all sofic graphs.

Proposition 3.1. *Let $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be super-additive and log-concave. Let T_n be as in Construction 2.1. Then the sequence (T_n, o_n) has a subsequence that converges in the Benjamini-Schramm sense to a unimodular random rooted tree of uniform growth g.*

This proves Theorem 2. We examine the structure of the Benjamini-Schramm limit $\mathcal{T}$:

Theorem 3 (Structure Theorem)**.**

(i) If $g = \Omega(r^{\alpha \log\log r})$, $\alpha > 1$, then $(\mathcal{T}, \omega)$ is a.s. an apocentric limit. In particular, it is 1-ended.

(ii) If $g = O(r^{\log\log r})$, then $(\mathcal{T}, \omega)$ is a.s. a balanced limit and it is a.s. 1-ended or 2-ended. In particular, if $\delta_n \neq 1$ eventually, then $(\mathcal{T}, \omega)$ is a.s. 1-ended and the probability for being isomorphic to any particular tree is 0.

The significance of the distinction worked out in Theorem 3 becomes more apparent with an example: the Benjamini-Schramm limit of (T_n, o_n) for a sufficiently fast growing function g (including intermediate) can be a.s. isomorphic to a single deterministic tree. Such a limit can be seen as deterministic trees with a randomly chosen root.

Example 3.2. Define recursively $\delta_1 := 1, \delta_2 := 2$ and $\delta_{n+1} := \delta_n \delta_{n-1}$. Hence, $\delta_n = 2^{F_n}$, for all $n \geq 0$, where F_n denotes the n-th Fibonacci number.

Let $T_n, n \geq 0$ be the sequence of trees according to Construction 2.1.

Then (T_n, o_n) has a subsequence that converges in the Benjamini-Schramm sense to a random rooted tree $(\mathcal{T}, \omega)$ of uniform volume growth.

From $|T_n| = (\delta_1 + 1) \cdots (\delta_n + 1)$ we have, for $r := 2^n$,

$$\tfrac{1}{2} D^{r^\alpha} \leq |T_n| \leq \tfrac{1}{2} r \cdot D^{r^\alpha}, \quad \text{with } D := 2^{\varphi^2/\sqrt{5}} \approx 2.251 \text{ and } \alpha := \log \varphi \approx 0.6942.$$

Here $\varphi \approx 1.618$ denotes the golden ratio. The growth is therefore *intermediate*.

We claim that $(\mathcal{T}, \omega)$ is a.s. isomorphic to a particular deterministic tree: from Theorem 3 we can see that $(\mathcal{T}, \omega)$ is an apocentric limit. We show that the T_n are highly symmetric in that any two apocentric copies $\tau, \tau' \prec_{n-1} T_n$ are in fact indistinguishable by symmetry. In consequence, there exists only one possible inclusion chain leading to an apocentric limit, and $\mathcal{T}$ is the unique tree obtained in this way.

Unimodular random rooted trees that are a.s. isomorphic to a unique tree of smaller uniform growth can be constructed by setting $\delta_{n+1} := \delta_n \delta_{n-1}$ for only some n, and $\delta_{n+1} := \delta_n$ otherwise.

In contrast, Benjamini-Schramm limits for g of growth below $r^{\log \log r}$ have measure zero on every countable set of trees, hence this approach cannot yield examples with uniform growth. It remains to ask whether this is an artifact of our construction or a general phenomenon.

4 Concluding remarks and open questions

4.1 Planar triangulations

Having established the existence of trees for various growth rates, we can use Construction 1.1 to conclude the existence of planar triangulations with the same range of growth behaviors. In fact, we can say more: previously known triangulations of polynomial growth are planar, but not necessarily triangulations of the plane, *i.e.,* they are not necessarily homeomorphic to $\mathbb{R}^2$. For this to be true, the tree T needs to be *1-ended*, which is the case *e.g.* for apocentric limits obtained from a sequence of Construction 2.1. Choosing a suitable metric on each triangle then also yields a Riemannian metric on $\mathbb{R}^2$ with the respective growth behavior.

4.2 Subgraphs of uniform growth

At the early stages of our research, the approach for constructing trees of uniform intermediate growth was to start from just any graph of intermediate growth (such as a Cayley graph of the Grigorchuk group [7]), and extract a spanning tree that inherits this growth in some way. Ironically, working out the details of this extraction led to an understanding of the desired trees that allowed us constructing them without a need for the ambient graph. Still, we ask:

Question 4.1. Given a graph G of uniform growth g, is there a spanning tree (or just any embedded tree) of the same uniform growth?

4.3 Beyond the construction

The unimodular random rooted trees of uniform volume growth constructed in Section 3 were obtained as Benjamini-Schramm limits of the sequence T_n. We found a threshold at growth $r^{\log\log r}$ and it remains open whether this is an artifact of our construction or whether it points to a fundamental phase change phenomenon in unimodular trees of uniform growth.

Question 4.2. To what extent are unimodular trees with growths on either side of the threshold $r^{\log\log r}$ structurally different?

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