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Rainbow spanning trees in uniformly coloured perturbed graphs

(Extended abstract)

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Abstract

We consider the problem of finding a copy of a rainbow spanning bounded-degree tree in the uniformly edge-coloured randomly perturbed graph.

Let G_0 be an n-vertex graph with minimum degree at least δn, and let T be a tree on n vertices with maximum degree at most d, where $\delta \in (0,1)$ and $d \geq 2$ are constants. We show that there exists $C = C(\delta, d) > 0$ such that, with high probability, if the edges of the union $G_0 \cup \mathbf{G}(n, C/n)$ are uniformly coloured with colours in $[n-1]$, then there is a rainbow copy of T.

Our result resolves in a strong form a conjecture of Aigner-Horev, Hefetz and Lahiri.

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1 Introduction

Given $\delta \in (0,1)$, we define $\mathcal{G}_{\delta,n}$ to be the family of graphs on $[n]$ with minimum degree at least δn, and we let $\mathbf{G}(n,p)$ be the binomial random graph on $[n]$ with edge probability p. One of the central themes in extremal combinatorics is understanding how large δ

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needs to be so that, for each $G \in \mathcal{G}_{\delta,n}$, G contains a copy of a given graph. Similarly, probabilistic combinatorics aims to determine how large p needs to be for a given graph to appear in $\mathbf{G}(n,p)$ with high probability[1]. As an interpolation between the two graph models, Bohman Frieze, and Martin [4] introduced the *perturbed graph* model. For a given $\delta \in (0,1)$, this is defined as $G_0 \cup \mathbf{G}(n,p)$ where $G_0 \in \mathcal{G}_{\delta,n}$, i.e. as the n-vertex graph on $[n]$ whose edge set is the union of the edges of G_0 and the edges of $\mathbf{G}(n,p)$. Since [4], there has been a sizeable body of research extending and adapting results from the extremal and the probabilistic to the perturbed setting.

Another flourishing trend is to investigate the emergence of rainbow structures in uniformly edge-coloured graphs. Given an edge-coloured graph G, a subgraph H of G is *rainbow* if each edge of H has a distinct colour. A graph G is *uniformly coloured* in a set of colours $\mathcal{C}$ if each edge of G gets a colour independently and uniformly at random from $\mathcal{C}$. For example, for $G = \mathbf{G}(n, \omega(1)/n)$ uniformly coloured in $\mathcal{C} = [n]$, Aigner-Horev, Hefetz and Lahiri [1] showed that with high probability G admits a rainbow copy of any fixed almost-spanning bounded-degree tree. Other instances of similar problems in random graphs can be found in [3, 5–7]. Here we consider rainbow spanning bounded-degree trees in uniformly coloured perturbed graphs.

Theorem 1.1. *Let $\delta \in (0,1)$ and let $d \geq 2$ be a positive integer. Then there exists $C > 0$ such that the following holds. Let G_0 be a graph on n vertices with minimum degree at least δn. Suppose that T is a tree on n vertices with maximum degree at most d, and that $G \sim G_0 \cup \mathbf{G}(n, C/n)$ is uniformly coloured in $[n-1]$. Then, with high probability, G contains a rainbow copy of T.*

Theorem 1.1 provides a rainbow variant of a result of Krivelevich, Kwan and Sudakov [12], who showed that, under the same assumptions, with high probability, $G_0 \cup \mathbf{G}(n, C/n)$ contains a copy of T.

Aigner-Horev, Hefetz and Lahiri [1] already considered the question of embedding rainbow spanning trees in uniformly coloured perturbed graphs, and they proved that the same conclusion holds when the edges are uniformly coloured with $(1+\varepsilon)n$ colours (for an arbitrary constant ε) and C/n is replaced by $\omega(1)/n$. Moreover, Theorem 1.1 proves in a strong form Conjecture 1.4 of [1].

In the next section, Section 2, we consider the problem of finding rainbow almost-spanning bounded-degree trees in uniformly coloured random graphs. In Section 3, we sketch how to prove our main result.

2 Almost-spanning rainbow trees in random graphs

The first ingredient in our proof is Theorem 2.1, which says that we can embed almost-spanning trees with bounded degree in a rainbow fashion in random subgraphs of uniformly coloured pseudorandom graphs. The reason we need to consider random subgraphs of

[1] Formally, we say that a sequence of events $(A_n)_{n\in\mathbb{N}}$ holds *with high probability* if $\mathbb{P}[A_n] \to 1$ as $n \to \infty$.

pseudorandom graphs, as opposed to standard random graphs, is explained in Section 3.1. We do not define what we mean by *pseudorandom* here.

For $p \in [0, 1]$, the *p-random subgraph* of a graph G, denoted by G_p, is the random graph resulting from sampling each edge of G independently with probability p.

Theorem 2.1. *Let $\varepsilon \in (0, 1)$ and let $d \geq 2$ be a positive integer. Then there exists $C > 0$ such that the following holds. Let T be a tree on $(1-\varepsilon)n$ vertices, with maximum degree d, let G be a pseudorandom graph on n vertices, and write $p = C/n$. Suppose that G_p is coloured uniformly in $[n]$. Then, with high probability, G_p contains a rainbow copy of T.*

Theorem 2.1 resolves Conjecture 1.2 of [1].

The proof of Theorem 2.1 uses two previous results. The first, due to Alon, Krivelevich and Sudakov [2, Thm. 1.4], says that sparse expander graphs contain a copy of every almost-spanning bounded-degree tree. Because with $p \geq C/n$, for a large constant $C > 0$, in the p-random subgraph of a pseudorandom graph sufficiently large subsets of vertices expand, this result from [2] implies the uncoloured version of Theorem 2.1.

The second result we use is a simple consequence of a general result of Ferber and Krivelevich [6, Thm. 1.2] for binomial random subgraphs of uniformly edge-coloured hypergraphs. This allows us to deduce Theorem 2.1 from its uncoloured version.

Theorem 2.2 (Consequence of [6, Thm. 1.2])**.** *Let $\varepsilon, p, q \in (0, 1)$ satisfy $q = \varepsilon^{-1}p$. Suppose that $\mathcal{H}$ is a collection of subgraphs of K_n with at most $(1-\varepsilon)n$ edges. Then*

$$\mathbb{P}\left[\begin{array}{c}\mathbf{G}(n,p)\ \textit{contains}\\ \textit{some}\ H \in \mathcal{H}\end{array}\right] \leq \mathbb{P}\left[\begin{array}{c}\textit{a uniformly edge-coloured}\ \mathbf{G}(n,q),\\ \textit{with colours in}\ [n],\ \textit{contains a rainbow}\ H \in \mathcal{H}\end{array}\right].$$

3 Rainbow spanning trees in randomly perturbed graphs

Let $G \sim G_0 \cup \mathbf{G}(n, C/n)$ and suppose G is uniformly coloured in $[n-1]$. Let T be the spanning tree of maximum degree at most d that we wish to embed in a rainbow fashion in G. Our proof splits into two cases, according to the structure of the tree T: when T has $\Omega(n)$ leaves; and when T has $\Omega(n)$ disjoint, not-too-short bare paths (where a *bare path* is a path whose interior vertices have degree 2 in T). An observation of Krivelevich [11] shows that each tree falls into at least one of these categories.

3.1 Embedding trees with long bare paths

Suppose that T has $\Omega(n)$ not-too-short disjoint bare paths. Consider r such paths of length ℓ (where $r = \Omega(n)$ and ℓ is a constant which is not too small), and denote the ends of the i-th path by x_i, y_i. Let F be the forest resulting from removing the interior vertices of these bare paths from T.

We will use Theorem 2.1 to embed F in G [2]. However, in order to be able to turn this into a rainbow embedding of T (by embedding a rainbow collection of r paths of length ℓ,

[2]Observe this still follows from Theorem 2.1 despite F being a forest. In fact we can find a rainbow embedding of the almost spanning tree which consists of F and the edges x_iy_i.

with the i-th path having endpoints x_i, y_i), we first prepare an absorbing structure, which is an adaptation of such a structure of Montgomery [13]. The building block of our absorber is given by the so-called (v, c)-*gadget*. These have been introduced by Gould, Kelly, Kühn and Osthus [8] in the context of random optimal proper colourings of the complete graph, and have already been used for perturbed graphs by the first two authors [9].

Given a vertex v and a colour c, a (v, c)-*gadget* $A_{v,c}$ is a graph on 11 vertices with the following property (the notation refers to Figure 1). $A_{v,c}$ contains two rainbow paths P and P' with the same end points, such that P uses all vertices in $A_{v,c}$ and has a c-coloured edge, and P' uses all vertices apart from v and all colours of P except for c.

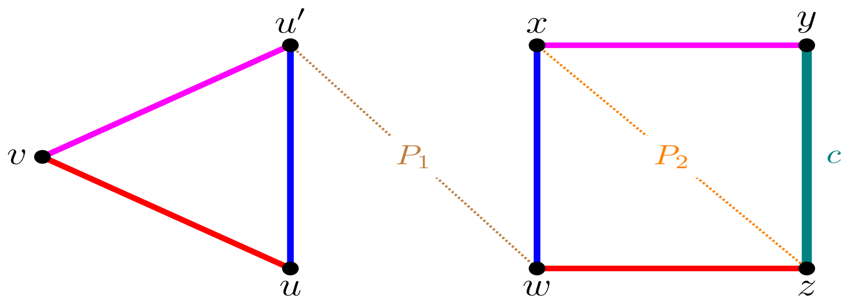

Figure 1: The (v, c)-gadget $A_{v,c}$, where the paths P_1 and P_2 have length three and are rainbow (with colours distinct from those already appearing). The path P (resp. P') is $uvu'P_1wxP_2zy$ (resp. $uu'P_1wzP_2xy$).

3.2 Embedding trees with many leaves

Suppose now that T has $\Omega(n)$ leaves. Roughly speaking, here is what we do. We first remove a constant proportion of the leaves, one leaf per parent, and embed the resulting almost-spanning tree in a rainbow fashion in G using Theorem 2.1. Completing this to a rainbow embedding of T amounts to finding a rainbow perfect matching between the removed leaves and their parents (since we removed one leaf for each parent), using all remaining colours. With some work, this follows from a forthcoming result of the authors [10], which in turn is an adaptation of a recent preprint of the first two authors [9].

Let L be a maximal collection of leaves with distinct parents. By the maximum degree assumption, $|L| = \Omega(n)$. Let L' be the collection of parents of the leaves in L, so $|L| = |L'|$. Let $T' = T \setminus L$. Let $\mathbf{G}_1 \sim \mathbf{G}(n, C/n)$ and colour $\mathbf{G}_1$ uniformly in $[n-1]$. By Theorem 2.1, with high probability, we can find a rainbow embedding of T' in $\mathbf{G}_1$[3]. Then, observe that the image of L' in V under the embedding, and the complement of $V(T')$ in the embedding, are distributed uniformly at random among all disjoint subsets of V of size $|L'|$.

[3] Theorem 2.1 applies when the number of vertices equals the number of colours, so formally it applies on a subgraph of $\mathbf{G}_1$ on $n-1$ vertices, which will be a binomial random graph with edge probability $C'/(n-1)$.

Draw a new copy of the random graph $\mathbf{G}_2 \sim \mathbf{G}(n, C/n)$. For each edge $e \in E(G_0) \cup E(\mathbf{G}_2)$, reveal whether its colour lies in $\mathcal{C}(T')$, the set of colours in the rainbow embedding of T'. Let G_0' be the subgraph of G_0 consisting of the edges which are disjoint from $E(\mathbf{G}_1)$ and have colours in $\mathcal{C}' := [n-1] \setminus \mathcal{C}(T')$. Then, from Chernoff's bound, it follows that $G_0'[L, L']$ has minimum degree $\Omega(n)$. Let $\mathbf{G}_2'$ be the subgraph of $\mathbf{G}_2$ whose edges are coloured in $\mathcal{C}'$. Then $\mathbf{G}_2'$ is a copy of the random graph $\mathbf{G}(n, C'/n)$, coloured uniformly in $\mathcal{C}'$, for an appropriate (but still large) constant C'.[4] Let $H = (G_0' \cup \mathbf{G}_2')[L, L']$. So H is a balanced bipartite graph, with bipartition $\{L, L'\}$, each of whose edges is coloured uniformly in $\mathcal{C}'$, a set of size $|L|$. It suffices to show that, with high probability, H has a rainbow perfect matching.

We now show that this reduces to finding a rainbow directed Hamilton cycle in a uniformly coloured directed perturbed graph. This can be proved as follows. Pick an arbitrary bijection $\pi : L' \to L$ and let D be the edge-coloured digraph on vertex set L with the following edges: for each $xy \in E(H)$, with $x \in L$ and $y \in L'$, add the directed edge $x\pi(y)$ and colour it by the colour of xy in H. It is straightforward to check that, if D has a rainbow directed Hamilton cycle, then H has a rainbow perfect matching. Indeed, suppose $x_1, \ldots, x_{|L|}$ is a rainbow Hamilton cycle in D. Then $x_1\pi^{-1}(x_2), x_2\pi^{-1}(x_3), \ldots, x_{|L|}\pi^{-1}(x_1)$ is a rainbow perfect matching in H.

It is also easy to check that D is distributed according to the *directed* perturbed model: this is the union of a digraph with linear minimum in- and out-degree, and $\mathbf{D}(n, p)$, the *random directed graph*, where each ordered pair of distinct vertices is an edge with probability p, independently. Moreover, D is uniformly coloured in $\mathcal{C}'$. The proof thus follows from the next theorem.

Theorem 3.1 ([10]). *Let $\delta \in (0, 1)$. Then there exists $C > 0$ such that the following holds. Let D_0 be a directed graph on vertex set $[n]$ with minimum in- and out-degree at least δn, and let $D \sim D_0 \cup \mathbf{D}(n, C/n)$ be uniformly coloured in $[n]$. Then, with high probability, D has a rainbow directed Hamilton cycle.*

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