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High-rank subtensors of high-rank tensors

(Extended abstract)

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Abstract

Let $d \geq 2$ be a positive integer. We show that for a class of notions R of rank for order-d tensors, which includes in particular the tensor rank, the slice rank and the partition rank, there exist functions $F_{d,R}$ and $G_{d,R}$ such that if an order-d tensor has R-rank at least $G_{d,R}(l)$ then we can restrict its entries to a product of sets $X_1 \times \cdots \times X_d$ such that the restriction has R-rank at least l and the sets $X_1, \ldots, X_d$ each have size at most $F_{d,R}(l)$. Furthermore, our proof methods allow us to show that under a very natural condition we can require the sets $X_1, \ldots, X_d$ to be pairwise disjoint.

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The results described below are proved and discussed further in the paper [5].

1 Main results

The last few years have seen a sequence of successes in using notions of ranks for higher-dimensional tensors to solve combinatorial problems. A central idea from the breakthrough solution to the cap-set problem by Ellenberg and Gijswijt [2], which was based on a technique of Croot, Lev, and Pach [1], was reformulated by Tao [11] in terms of the notion of slice rank for tensors, leading to what is now known as the slice rank polynomial method. The slice rank was further studied by Sawin and Tao [10], and bounds shown there on

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the slice rank involving orderings on the coordinates were later used by Sauermann [9] to prove under suitable conditions the existence of solutions with pairwise distinct variables to systems of equations in subsets of $\mathbb{F}_p^n$ that are not exponentially sparse. Another fruitful generalisation of the idea underlying the slice rank has been the partition rank, which was defined by Naslund [8] in order to prove a polynomial upper bound on the size of subsets of $\mathbb{F}_{p^r}^n$ not containing any k-right corners (with p a prime integer and $r \geq 1$ a positive integer) and very recently used again by Naslund [7] to prove exponential lower bounds on the chromatic number of $\mathbb{R}^n$ with multiple forbidden distances.

We will focus on high-rank subtensors of tensors: it is a standard fact from linear algebra that if A is a matrix of rank k then A has a $k \times k$ submatrix with rank k, and we will study here the extent to which this statement can be generalised to notions of rank for higher-order tensors, in particular to the tensor rank, to the slice rank and to the partition rank. The results that we obtain in this direction as well as the methods that we use in their proofs will also allow us to prove that under a very natural assumption we can find a subtensor such that the coordinates take values in pairwise disjoint sets. As we explain in a few paragraphs, the formulation of this second result also arises naturally as an analogue of the standard inequality that every oriented graph has a bipartition such that at least a quarter of the edges go from the first part to the second.

We now define the relevant notions of higher-dimensional ranks for tensors and state our main theorems.

Definition 1. *Let $d \geq 2$ be a positive integer and let $\mathbb{F}$ be a field. An* order-d tensor *over $\mathbb{F}$ is a function $T : Q_1 \times \cdots \times Q_d \to \mathbb{F}$ for some finite subsets $Q_1, \ldots, Q_d$ of $\mathbb{N}$.*

Throughout we shall use the following notation. We write $\mathbb{F}$ for an arbitrary field, and all our statements will hold uniformly in $\mathbb{F}$. If $d \geq 2$ is a positive integer, then $Q_1, \ldots, Q_d$ will always stand for finite subsets of $\mathbb{N}$. Given an order-d tensor $T : Q_1 \times \cdots \times Q_d \to \mathbb{F}$ and subsets $X_1 \subset Q_1, \ldots, X_d \subset Q_d$, we shall write $T(X_1 \times \cdots \times X_d)$ for the restriction $X_1 \times \cdots \times X_d \to \mathbb{F}$ of T. For each positive integer n we write $[n]$ for the set $\{1, 2, \ldots, n\}$. Given $x \in Q_1 \times \cdots \times Q_d$, and $I \subset [d]$, we write $x(I)$ for the restriction $(x_\alpha : \alpha \in I)$ of x to its coordinates in I.

Definition 2. *Let $d \geq 2$ be a positive integer, and let T be an order-d tensor. We say that T has* tensor rank at most 1 *if there exist functions $a_\alpha : Q_\alpha \to \mathbb{F}$ for each $\alpha \in [d]$ such that*

$$T(x_1, \ldots, x_d) = a_1(x_1) \ldots a_d(x_d)$$

for every $(x_1, \ldots, x_d) \in Q_1 \times \cdots \times Q_d$.

We say that T has slice rank at most 1 *if there exist $\alpha \in [d]$ and functions $a : Q_\alpha \to \mathbb{F}$ and $b : \prod_{\alpha' \in [d], \alpha' \neq \alpha} Q_{\alpha'} \to \mathbb{F}$ such that we can write*

$$T(x_1, \ldots, x_d) = a(x_\alpha) b(x_1, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_d)$$

for every $(x_1, \ldots, x_d) \in Q_1 \times \cdots \times Q_d$.

We say that T has partition rank at most 1 *if there exist a bipartition $\{I, J\}$ of $[d]$ with I, J both non-empty and functions $a : \prod_{\alpha \in I} Q_\alpha \to \mathbb{F}$ and $b : \prod_{\alpha \in J} Q_\alpha \to \mathbb{F}$ such that we can write*

$$T(x_1, \dots, x_d) = a(x(I))b(x(J))$$

for every $(x_1, \dots, x_d) \in Q_1 \times \cdots \times Q_d$.

We say that the tensor rank *(resp.* slice rank*, resp.* partition rank*) of T is the smallest nonnegative integer k such that there exist tensors $T_1, \dots, T_k$ each of tensor rank at most 1 (resp. slice rank at most 1, resp. partition rank at most 1) and such that $T = T_1 + \cdots + T_k$. We denote by $\operatorname{tr} T$ the tensor rank of T, by $\operatorname{sr} T$ the slice rank of T, and by $\operatorname{pr} T$ the partition rank of T.*

We will begin by showing the fact that every matrix of rank k has a $k \times k$ subtensor with rank k generalises in the best way one could hope for to the tensor rank for all $d \geq 2$: every order-d tensor T with tensor rank k has a $k \times k \times \cdots \times k$ (d times) subtensor with tensor rank k. However, that becomes false for the order-3 slice rank: we thank Timothy Gowers for constructing a counterexample. It will nonetheless be true that if an order-3 tensor is such that all its subtensors with size at most $48l^3$ have slice rank at most l then the whole tensor has slice rank at most $51l^3$. Finally we will show that such an asymptotic subtensors property holds for the slice and partition rank for all $d \geq 2$ as well as for a more general class of notions of rank which we now define before stating this asymptotic result.

Definition 3. *Let $d \geq 2$ be a positive integer, and let R be a non-empty family of partitions of $[d]$. We say that an order-d tensor T has R-*rank at most 1 *if there exist a partition $P \in R$ and for each $I \in P$ a function $a_I : \prod_{\alpha \in I} Q_\alpha \to \mathbb{F}$ such that we can write*

$$T(x_1, \dots, x_d) = \prod_{I \in P} a_I(x(I))$$

*for every $(x_1, \dots, x_d) \in Q_1 \times \cdots \times Q_d$. We say that the R-*rank *of T is the smallest nonnegative integer k such that there exist order-d tensors $T_1, \dots, T_k$ with R-rank at most 1 such that $T = T_1 + \cdots + T_k$.*

We will denote by $R\mathrm{rk}\, T$ the R-rank of T. We can check that for every $d \geq 2$, the R-rank specialises to the tensor rank, to the slice rank, and to the partition rank.

We are now in a position to state our first main theorem.

Theorem 4. *Let $d \geq 2$ be a positive integer, and let R be a non-empty family of partitions of $[d]$. There exist functions $F_{d,R} : \mathbb{N} \to \mathbb{N}$ and $G_{d,R} : \mathbb{N} \to \mathbb{N}$ such that if T is an order-d tensor with $R\mathrm{rk}\, T \geq G_{d,R}(l)$ then there exist $X_1 \subset Q_1, \dots, X_d \subset Q_d$ each with size at most $F_{d,R}(l)$ such that $R\mathrm{rk}\, T(X_1 \times \cdots \times X_d) \geq l$.*

Another independent starting point is the following standard statement.

Proposition 5. *Let G be an oriented graph with vertex set V. There exists an ordered bipartition (X, Y) of V such that the number of edges $(u, v) \in X \times Y$ of G is at least a quarter of the total number of edges of G.*

This statement can be seen to be equivalent to the following: given a matrix $A : [n] \times [n] \to \mathbb{F}$ there exist disjoint subsets X, Y of $[n]$ such that the restriction $A(X \times Y)$ has at least a quarter as many support elements as A has outside the diagonal. A first step will be to obtain an analogue of this statement for ranks of matrices. We thank Lisa Sauermann for a sketch that led to the proof of that statement. We will then generalise this analogue to higher-order tensors. We note that Proposition 5 and its generalisation to uniform hypergraphs will themselves be involved in the proof of the general higher-order tensor case.

Let E be the set of points $(x_1, \dots, x_d) \in Q_1 \times \dots \times Q_d$ that do *not* have pairwise distinct coordinates. The following definition will be central to our second main result.

Definition 6. *Let $d \geq 2$ be a positive integer, let R be a non-empty family of partitions of $[d]$. For $T : Q_1 \times \dots \times Q_d \to \mathbb{F}$ an order-d tensor we define the* essential R-rank

$$\mathrm{e}R\mathrm{rk}\, T = \min_V R\mathrm{rk}(T + V)$$

where the minimum is taken over all order-d tensors $V : Q_1 \times \dots \times Q_d \to \mathbb{F}$ with support contained inside E, and the disjoint R-rank

$$\mathrm{d}R\mathrm{rk}\, T = \max_{X_1,\dots,X_d} R\mathrm{rk}(T(X_1 \times \dots \times X_d))$$

where the maximum is taken over all $X_1 \subset Q_1, \dots, X_d \subset Q_d$ with $X_1, \dots, X_d$ pairwise disjoint.

It seems worthwhile to compare the essential R-rank with the disjoint R-rank, as it is straightforward to show that a tensor has essential R-rank equal to 0 if and only if it has disjoint R-rank equal to 0: the corresponding tensors are the tensors supported inside E. Moreover, we can show that the disjoint R-rank is at most the essential R-rank.

Lemma 7. *Let $d \geq 2$ be a positive integer, and let R be a non-empty family of partitions of $[d]$. For every order d tensor $T : Q_1 \times \dots \times Q_d \to \mathbb{F}$ we have*

$$\mathrm{d}R\mathrm{rk}\, T \leq \mathrm{e}R\mathrm{rk}\, T.$$

Our second main result is a weak converse to this last inequality.

Theorem 8. *Let $d \geq 2$ be a positive integer, and let R be a non-empty family of partitions of $[d]$. There exists a function $G'_{d,R} : \mathbb{N} \to \mathbb{N}$ such that if T is an order-d tensor such that $\mathrm{e}R\mathrm{rk}\, T \geq G'_{d,R}(l)$ then we have $\mathrm{d}R\mathrm{rk}\, T \geq l$.*

Theorem 8 is also an essential ingredient to the proof of the main result of the paper [3], where in joint work with Timothy Gowers we generalise a theorem of Green and Tao ([4], Theorem 1.7) on the approximate equidistribution of polynomials with high rank over finite prime fields to the case where the variables are chosen (uniformly and independently) at random in an arbitrary non-empty subset of the field rather than in the whole field. However, we will not focus on this application.

The methods involved in our proofs of Theorem 4 and of Theorem 8 are similar in several ways: those that we will use to prove the latter can be viewed as a moderate complication of those that we will use to prove the former.

2 Proof example

As a simple representative example of our proof techniques, let us explain how we show Theorem 4 in the case of the order-3 slice rank, assuming that it is already proved in the case of the order-3 tensor rank. We begin by proving a lemma showing that having a large separated set of slices guarantees a high slice rank. For $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ and $x \in Q_1$ we write $T_x : Q_2 \times Q_3 \to \mathbb{F}$ for the matrix defined by $T_x(y, z) = T(x, y, z)$, and similarly define the notations T_y and T_z.

Lemma 9. *Let $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ be an order-3 tensor, and $l \geq 1$ be an integer. If there exist $x_1 \dots, x_l \in Q_1$ such that*

$$\operatorname{rk}(\sum_{i=1}^{l} a_i T_{x_i}) \geq l$$

for every $(a_1, \dots, a_l) \in \mathbb{F}^l \setminus \{0\}$, then $\operatorname{sr} T \geq l$.

We next show a partial converse to the inequality $\operatorname{sr} T \leq \operatorname{tr} T$ which holds in the situation where all slices of T of all three kinds have bounded rank.

Lemma 10. *Let $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ be an order-3 tensor. Let $m \geq 1$ be a positive integer. Assume that for all $x \in Q_1, y \in Q_2, z \in Q_3$ we have $\operatorname{rk} T_x, \operatorname{rk} T_y, \operatorname{rk} T_z \leq m$. Then $\operatorname{tr} T \leq m(\operatorname{sr} T)^2$.*

We are now ready to finish the proof.

Proposition 11. *Let $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ be an order-3 tensor, and let $l \geq 1$ be a positive integer. If $\operatorname{sr} T \geq 51l^3$ then there exist $X \subset Q_1$, $Y \subset Q_2$, $Z \subset Q_3$ with size at most $48l^3$ such that $\operatorname{sr} T(X \times Y \times Z) \geq l$.*

Let $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ be an order-3 tensor. If T satisfies the assumption of Lemma 9 then we can conclude using a multidimensional version of the standard statement on submatrices. If T satisfies the assumption of Lemma 10 then we conclude by reducing to the tensor rank. Furthermore, these two lemmas can be viewed to some extent as representing two extreme cases, to which we can always reduce: if T is an order-3 tensor with high slice rank but which is not in the first situation, then we can always decompose it as a sum $S + U$ where S has bounded slice rank and U is in the second situation, a decomposition which hence allows us to prove Proposition 11 for all order-3 tensors.

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