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Perfect-matching covers of cubic graphs with colouring defect 3

(Extended abstract)

Ján Karabáš* Edita Máčajová† Roman Nedela‡ Martin Škoviera§

Abstract

The colouring defect of a cubic graph is the smallest number of edges left uncovered by any set of three perfect matchings. While 3-edge-colourable graphs have defect 0, those that cannot be 3-edge-coloured have defect at least 3. We show that every bridgeless cubic graph with defect 3 can have its edges covered with at most five perfect matchings, which verifies a long-standing conjecture of Berge for this class of graphs. Moreover, we determine the extremal graphs.

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1 Introduction

A strong form of Petersen's Perfect Matching Theorem [15] states that each edge of a bridgeless cubic graph G is contained in a perfect matching. The minimum number of perfect matchings needed to cover all the edges of G is its *perfect matching index*, denoted by $\pi(G)$. In 1970's, Berge conjectured (unpublished, see [3, 10, 16]) that $\pi(G) \leq 5$ for

*Faculty of Natural Sciences, Matej Bel University, Banská Bystrica, SK and Mathematical Institute of Slovak Academy of Sciences, Banská Bystrica, SK. E-mail: jan.karabas@umb.sk. Supported by grants APVV-19-0308 and VEGA 2/0078/20.

†Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava, SK. E-mail: macajova@dcs.fmph.uniba.sk. Supported by grants APVV-19-0308 and VEGA 1/0743/21.

‡Faculty of Applied Sciences, University of West Bohemia, Pilsen, CZ and Mathematical Institute of Slovak Academy of Sciences, Banská Bystrica, SK. E-mail: nedela@savbb.sk. Supported by grants APVV-19-0308 and VEGA 2/0078/20.

§Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava, SK. E-mail: skoviera@dcs.fmph.uniba.sk. Supported grants APVV-19-0308 and VEGA 1/0727/22.

every bridgeless cubic graph G. After more than 50 years, this conjecture remains widely open. In fact, very little is known unless the graph in question has a very specific structure, see for example [1, 4, 5, 9]).

In this paper we investigate perfect-matching covers of bridgeless cubic graphs that are close to 3-edge-colourable cubic graphs. If a cubic graph G can be 3-edge-coloured, then obviously $\pi(G) = 3$, and $\pi(G) \geq 4$ otherwise. If G cannot be 3-edge-coloured, then any set $\{M_1, M_2, M_3\}$ of three perfect matchings of G leaves some edges uncovered. The minimum number of uncovered edges is the *colouring defect* of G, denoted by $\mathrm{df}(G)$. This concept was introduced and extensively studied by Steffen et al. in [8, 17]. Together with oddness, resistance, perfect matching index, and other similar invariants it serves as one of measures of uncolourability of cubic graphs [2].

Steffen [17] showed that the colouring defect of every non-3-edge-colourable cubic graph (henceforth just *defect*, for short) is at least 3. Cubic graphs with defect 3 thus constitute a class of cubic graphs that is in a certain sense closest to 3-edge-colourable graphs. The purpose of this paper is to show that Berge's conjecture holds for this class of cubic graphs and to characterise the extremal graphs where five perfect matchings are actually necessary. Our main result reads as follows.

Theorem 1.1. *Every bridgeless cubic graph G of defect 3 can have its edges covered with at most five perfect matchings; that is, $4 \leq \pi(G) \leq 5$. If G is 3-connected, then $\pi(G) = 5$ if and only if G arises from the Petersen graph by inflating any number of vertices of a fixed vertex-star (possibly zero) by quasi-bipartite cubic graphs in a correct way.*

For cubic graphs with defect 3 this result significantly improves the result of Steffen [17, Theorem 2.14] which states that every cyclically 4-edge-connected cubic graph with defect 3 or 4 satisfies Berge's conjecture.

2 Auxiliary results

The proof of Theorem 1.1 will be executed in several steps and will use a number of tools. One of key ingredients, applied several times and at various stages of the proof, is the following theorem which explores 6-edge-cuts in cubic graphs. Given a subgraph H of a graph G, let $\delta_G(H)$ denote the edge-cut comprising all edges with exactly one end in H.

Theorem 2.1. *Let G be a bridgeless cubic graph and let $H \subseteq G$ be a subgraph with $|\delta_G(H)| = 6$. Then H has a perfect matching, or else H contains an independent set S of trivalent vertices such that*

(i) *the number of components of $H - S$ equals $|S| + 2$, and*

(ii) *every component L of $H - S$ has $|\delta_G(L)| = 3$.*

A bridgeless cubic graph Q will be called *quasi-bipartite* if it contains an independent set of vertices U such that the graph obtained by the contraction of each component of

$Q - U$ to a vertex is a cubic bipartite graph where U is one of the partite sets. Roughly speaking, a quasi-bipartite cubic graph arises from a bipartite cubic graph by inflating certain vertices in one of the partite sets to larger subgraphs, while preserving the edges between the partite sets. The previous theorem thus implies that if we add two new vertices u and v to H and create a cubic graph H^+ from H by attaching the edges of $\delta_G(H)$ to u and v, then H^+ becomes a quasi-bipartite with the independent set $U = S \cup \{u, v\}$.

The second auxiliary result is also related to bipartite graphs. A cubic graph G is said to be *almost bipartite* if it is bridgeless, not bipartite, and contains two edges e and f such that $G - \{e, f\}$ is a bipartite graph. The edges e and f are the *surplus edges* of G. Observe that if a cubic graph G is almost bipartite, then it has a component such that e connects vertices within one partite set and f connects vertices within the other partite set. Moreover, it can be shown that G has a perfect matching that contains both surplus edges. As a consequence, we obtain the following.

Theorem 2.2. *Every almost bipartite cubic graph is* 3*-edge-colourable.*

The *bipartite index* of a graph G is defined to be the smallest number of edges that must be deleted in order to make the graph bipartite. The previous theorem implies that every bridgeless cubic graph with bipartite index at most 2 is 3-edge-colourable. On the other hand, there exist infinitely many snarks whose bipartite index equals 3, for example the Isaacs flower snarks [6]. In this sense, Theorem 2.2 is best possible.

3 Berge covers for cubic graphs of defect 3

A *Berge cover* of a cubic graph G is a collection of five perfect matchings that cover all the edges of G. To find such a cover for a graph of defect 3 we employ a structure created by three perfect matchings. For a bridgeless cubic graph G we define an *optimal* 3*-array* of perfect matchings to be any set $\mathcal{M} = \{M_1, M_2, M_3\}$ of three perfect matchings such that the number of edges not covered by $M_1 \cup M_2 \cup M_3$ equals the defect of G. The *core* of $\mathcal{M}$ is the subgraph of G induced by the set of all edges that are not simply covered by $\mathcal{M}$. It is not difficult to see that if $\mathrm{df}(G) = 3$, then the core of $\mathcal{M}$ is a chordless hexagon which alternates the uncovered edges with the doubly covered ones [17]. If G is the Petersen graph, then any hexagon can be taken as the core of a suitable optimal 3-array. In particular, the defect of the Petersen graph equals 3.

To prove the first statement of Theorem 1.1 we show that every optimal 3-array $\mathcal{M}$ for a cubic graph G with $\mathrm{df}(G) = 3$ extends to a Berge cover. The key step towards the proof is the next lemma. At the crucial moment of its proof we apply Theorem 2.1 to the 6-edge-cut $\delta_G(W)$ where W is a suitable path of length 3 lying in the core of $\mathcal{M}$.

Lemma 3.1. *Let G be a bridgeless cubic graph of defect* 3 *and let $\mathcal{M}$ be an optimal* 3*-array of perfect matchings of G. Then G has a fourth perfect matching which covers at least two of the three edges left uncovered by $\mathcal{M}$.*

With the help of Lemme 3.1 we can prove the following.

Theorem 3.2. *Every bridgeless cubic graph with defect* 3 *has a Berge cover.*

Proof. Assume that $\mathrm{df}(G) = 3$, and let $\mathcal{M}$ be an arbitrary optimal array for G. Let M_4 be a perfect matching guaranteed by Lemma 3.1, which covers at least two of the uncovered edges. Since G has perfect matching that covers any preassigned edge, we can take a perfect matching M_5 that covers the third uncovered edge. Clearly, $\mathcal{M} \cup \{M_4, M_5\}$ is a Berge cover of G. □

4 Cyclically 4-edge-connected graphs

Our main result restricted to cyclically 4-edge-connected graphs reads as follows.

Theorem 4.1. *Let G be a cyclically* 4*-edge-connected cubic graph with defect* 3*. Then* $\pi(G) = 4$*, unless G is the Petersen graph.*

Proof (sketch). We prove that if $\pi(G) \geq 5$, then G is the Petersen graph. Take an optimal 3-array $\mathcal{M}$ for G whose core is a 6-cycle $C = (v_0 v_1 \dots v_5)$, and set $H = G - V(C)$. If H had a perfect matching, we could extend it to a perfect matching M_4 of the entire G in such a way that $\mathcal{M} \cup \{M_4\}$ covers all the edges of G, implying that $\pi(G) = 4$. Therefore H has no perfect matching, and we can apply Theorem 2.1 to the edge-cut $\delta_G(H)$. Let $S \subseteq V(H)$ be the independent set of trivalent vertices stated in Theorem 2.1. Then each component of $H - S$ is a single vertex, because G is 4-edge-connected. It follows that H is bipartite, and therefore 3-edge-colourable.

We now investigate the 6-tuples of colours on $\delta_G(H)$ induced by 3-edge-colourings of H, ordered cyclically around C. It is easy to see that all three colours must always occur, otherwise the missing colour could be extended to a perfect matching M_4 of G, yielding a contradiction as before. There remain 15 colouring types for $\delta_G(H)$ of which 7 are excluded because they would enable a 3-edge-colouring of G.

For $i \in \{0, \dots, 5\}$ let u_i be the neighbour of v_i lying in H. We claim that $u_i = u_j$ whenever $j \equiv i + 3 \pmod 6$. If $u_i = u_j$, then indeed $j \equiv i + 3 \pmod 6$, otherwise G would have a triangle or a quadrilateral intersecting C. The former possibility cannot occur due to cyclic connectivity. In the latter case, the quadrilateral would share two edges with C, in which case $\mathcal{M}$ could be modified to a 3-edge-colouring of G, a contradiction. Suppose that there exist vertices u_i and u_j such that $u_i \neq u_j$ and $j \equiv i + 3 \pmod 6$, say $u_2 \neq u_5$. Create a cubic graph $H^\sharp$ from H as follows: add two new vertices s and t, connect them between themselves and to $\{u_0, u_1, u_3, u_4\}$, and finally join u_2 to u_5. This can be done in such a way that no 3-edge-colouring of H extends to $H^\sharp$, implying that $H^\sharp$ is not 3-edge-colourable. However, $H^\sharp$ is almost bipartite, which contradicts Theorem 2.2. Therefore $u_0 = u_3$, $u_1 = u_4$, and $u_2 = u_5$. It follows that $\delta_G(C \cup \{u_0, u_1, u_2\})$ is a 3-edge-cut, which must be trivial due to cyclic connectivity. Hence G has $6 + 3 + 1$ vertices, and this means that G is the Petersen graph. □

5 General case of Theorem 1.1

To move away from cyclically 4-edge-connected graphs we modify the classical method of snark reduction to cubic graphs of defect 3. By a *snark* we mean a 2-connected cubic graph that admits no 3-edge-colouring. A snark is *nontrivial* if it is cyclically 4-edge-connected with girth at least 5. It is well known that every snark can be transformed to a nontrivial snark by a sequence of certain simple reductions (like contracting a triangle). Performing a reduction of a snark G means to identify an edge-cut R in G whose removal leaves a component H which is not 3-edge-colourable. By adding a small number of vertices or edges it is possible to extend H to a snark G', a *reduction* of G along R.

A reduction of a snark G with defect 3 to a nontrivial snark G' of defect 3 may not always be possible. Such a situation occurs, for example, when G contains an *essential triangle*, one whose contraction produces a snark with defect greater than 3. It can be shown that the increase of defect by contracting an essential triangle can be arbitrarily large. Nevertheless, a snark with defect 3 can have at most one essential triangle, and if so, then it is the only obstruction to reduction.

Theorem 5.1. *Every snark G with $\mathrm{df}(G) = 3$ admits a reduction to a snark G' with $\mathrm{df}(G') = 3$ such that either G' is nontrivial or G' arises from a nontrivial snark K with $\mathrm{df}(K) \geq 4$ by inflating a vertex to a triangle; the triangle is essential in both G and G'.*

The proof of this theorem is quite involved and requires a careful analysis of Fano flows associated with 3-arrays (for the definition of a Fano flow see [7]).

Reductions can be conveniently handled with the help of two well-known operations. Let G and H be cubic graphs with distinguished edges e and f, respectively. We define a 2-*sum* $G \oplus_2 H$ to be a cubic graph obtained by deleting e and f and connecting the 2-valent vertices of G to those of H. If instead of distinguished edges we have distinguished vertices u and v of G and H, respectively, we can similarly define a 3-sum $G \oplus_3 H$. Note that $G \oplus_3 H$ can be regarded as being obtained from G by *inflating* the vertex u to $H - v$.

A cubic graph G containing a cycle-separating 2-edge-cut or 3-edge-cut can be expressed as $G \oplus_2 H$ or $G \oplus_3 H$ uniquely, only depending on the chosen edge-cut. It is easy to see that if two 2-cuts or 3-cuts intersect, the result of decomposition does not depend on the order in which the cuts are taken. As a consequence, we have the following.

Theorem 5.2. *Every 2-connected cubic graph G admits a decomposition into a collection $\{G_1, \ldots, G_m\}$ of cyclically 4-edge-connected cubic graphs such that G can be reconstructed from them by a repeated application of 2-sums and 3-sums. Moreover, this collection is unique up to ordering and isomorphism.*

The first step in the proof of the general case of Theorem 1.1 is to show that, somewhat surprisingly, cubic graphs with defect 3 containing an essential triangle behave nicely.

Theorem 5.3. *If a cubic graph G with defect 3 has an essential triangle, then $\pi(G) = 4$.*

Proof (sketch). Let T be an essential triangle of G and let $\mathcal{M}$ be an optimal 3-array for G with hexagonal core C. Let w be the unique neighbour of T not lying on C. First assume that G/T is cyclically 4-edge-connected. We claim that G has a perfect matching M_4 such that $\mathcal{M} \cup \{M_4\}$ covers all the edges of G. If not, we apply Theorem 2.1 to the 6-cut $\delta_G(C \cup T \cup \{w\})$ and with the help of Theorem 2.2 we derive a contradiction in a similar manner as in the proof of Theorem 4.1. If G is not cyclically 4-edge-connected, we contract T to a vertex t and decompose G/T to $K_1, \ldots, K_m$ according to Theorem 5.2. Exactly one K_i, say K_1, contains the vertex t. We inflate t back to T, transforming K_1 to a graph L. Note that C survives the decomposition of G intact, so T is an essential triangle of L. As L/T is cyclically 4-edge-connected, Theorem 4.1 implies that $\pi(L/T) = 4$ or L/T is the Petersen graph. In both cases L has a cover with four perfect matchings. By using 2-sums and 3-sums this cover can be extended to a cover of the entire G. □

Let G and H be 2-connected cubic graphs where H is quasi-bipartite with independent set U. We say that a 3-sum $G \oplus_3 H$ is *correct* if the distinguished vertex of H forms a trivial component of $H - U$. Note that the result of a correct 3-sum is again quasi-bipartite.

Theorem 5.4. *Let G and H be 2-connected cubic graphs where $\pi(G) \geq 5$ and H is 3-edge-colourable. Then $\pi(G \oplus_3 H) \geq 5$ if and only if H is quasi-bipartite and the 3-sum is correct.*

Our intention is to characterise all 2-connected cubic graphs G with $\mathrm{df}(G) = 3$ and $\pi(G) = 5$. If G has a 2-edge-cut, then G can be expressed as $G' \oplus_2 H$. Since no hexagonal core can intersect a 2-edge-cut, the core stays within one summand, say G'. We conclude that $\mathrm{df}(G') = 3$, $\pi(G') = 5$, and that H is 3-edge-colourable. It follows that it is enough to characterise 3-connected cubic graphs with $\mathrm{df}(G) = 3$ and $\pi(G) = 5$. This is done in the next theorem, whose proof concludes that of Theorem 1.1.

Theorem 5.5. *Let G be a 3-connected cubic graph with $\mathrm{df}(G) = 3$. Then $\pi(G) = 5$ if and only if G arises from the Petersen graph by inflating any number of vertices of a fixed vertex-star by quasi-bipartite cubic graphs in a correct way.*

Proof (sketch). Assume that $\pi(G) = 5$. The statement is clearly true if G is cyclically 4-edge-connected, so we may assume that $H_0 = G$ can be expressed as a 3-sum $H_1 \oplus_3 H_1'$. By Theorem 5.3, G has no essential triangle, so every hexagonal core of G survives in one of the summands, say H_1. By Theorem 5.4, $\pi(H_1) = 5$, H_1' is quasi-bipartite, and the 3-sum is correct. We now continue with the decomposition by applying Theorem 5.4 to H_1, and so forth. Eventually, we obtain a collection $\{G_1, \ldots, G_m\}$ of cyclically 4-edge-connected cubic graphs exactly one of which, say G_1, is a snark, which has $\mathrm{df}(G_1) = 3$ and $\pi(G_2) = 5$. By Theorem 4.1, G_1 is the Petersen graph. It means that G arises from G_1 by a repeated correct 3-sum with a number of quasi-bipartite graphs. Since a fixed hexagon $C \subseteq G_1$ must survive the summation as a core, only the four vertices of $G - V(C)$, forming a vertex-star complementary to C, are eligible as distinguished vertices for 3-sums. Thus G has the structure which is described in Theorem 1.1. The reverse implication proceeds along similar lines. □

6 Final remarks

This paper summarises results presented in several papers at various stages of writing. Full proofs of Theorems 2.1-2.2, Theorem 3.2, and Theorem 4.1 can be found in [12], which is available on arXiv. Theorem 5.1 and the fact that the contraction of an essential triangle can increase defect arbitrarily are proved in [13]. The latter result heavily depends on results proved in [14, Theorems 5.1-5.2]. Finally, Theorems 5.3-5.5 and Theorem 1.1 will be proved in [11].

References

[1] F. Chen and G. Fan. Fulkerson-covers of hypohamiltonian graphs. *Discrete Appl. Math.*, 186:66–73, 2015.

[2] M. A. Fiol, G. Mazzuoccolo, and E. Steffen. Measures of edge-uncolorability of cubic graphs. *Electron. J. Combin.*, 25(P4.54), 2018.

[3] J.-L. Fouquet and J.-M. Vanherpe. On Fulkerson conjecture. *Discuss. Math. Graph Theory*, 31:253–272, 2011.

[4] R.-X. Hao, X. Wang J. Niu, C.-Q. Zhang, and T. Zhang. A note on Berge-Fulkerson coloring. *Discrete Math.*, 309:4235–4240, 2009.

[5] R.-X. Hao, C.-Q. Zhang, and T. Zheng. Berge-Fulkerson coloring for $C_{(8)}$-linked graphs. *J. Graph Theory*, 88:46–60, 2018.

[6] R. Isaacs. Infinite families of non-trivial trivalent graphs which are not Tait colorable. *Amer. Math. Monthly*, 82:221–239, 1975.

[7] L. Jin, G. Mazzuoccolo, and E. Steffen. Cores, joins and the Fano-flow conjectures. *Discuss. Math. Graph Theory*, 38:165–175, 2018.

[8] L. Jin and E. Steffen. Petersen cores and the oddness of cubic graphs. *J. Graph Theory*, 84:109–120, 2017.

[9] S. Liu, R.-X. Hao, and C.-Q. Zhang. Rotation snark, Berge-Fulkerson conjecture and Catlin's 4-flow reduction. *Appl. Math. Comput.*, 410:126441, 2021.

[10] G. Mazzuoccolo. The equivalence of two conjectures of Berge and Fulkerson. *J. Graph Theory*, 68:125–128, 2011.

[11] J. Karabáš, E. Máčajová, R. Nedela, and M. Škoviera. Berge covers of cubic graphs with colouring defect 3. in preparation, 2022.

[12] J. Karabáš, E. Máčajová, R. Nedela, and M. Škoviera. Berge's conjecture for cubic graphs with small colouring defect. `arXiv:2210.13234 [math.CO]`, 2022.

[13] J. Karabáš, E. Máčajová, R. Nedela, and M. Škoviera. Cubic graphs with colouring defect 3. manuscript, 2022.

[14] J. Karabáš, E. Máčajová, R. Nedela, and M. Škoviera. Girth, oddness, and colouring defect of snarks. *Discrete Math.*, 345:113040, 2022.

[15] T. Schönberger. Ein Beweis des Petersenschen Graphensatzes. *Acta Litt. Sci. Szeged*, 7:51–57, 1934.

[16] P. D. Seymour. On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte. *London. Math. Soc.*, 38:423–460, 1979.

[17] E. Steffen. 1-Factor and cycle covers of cubic graphs. *J. Graph Theory*, 78:195–206, 2015.