SEMI-ALGEBRAIC AND SEMI-LINEAR RAMSEY NUMBERS

(EXTENDED ABSTRACT)

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Abstract

An r-uniform hypergraph H is *semi-algebraic* of complexity $\mathbf{t} = (d, D, m)$ if the vertices of H correspond to points in $\mathbb{R}^d$, and the edges of H are determined by the sign-pattern of m degree-D polynomials. Semi-algebraic hypergraphs of bounded complexity provide a general framework for studying geometrically defined hypergraphs.

The much-studied *semi-algebraic Ramsey number* $R_r^{\mathbf{t}}(s, n)$ denotes the smallest N such that every r-uniform semi-algebraic hypergraph of complexity $\mathbf{t}$ on N vertices contains either a clique of size s, or an independent set of size n. Conlon, Fox, Pach, Sudakov and Suk proved that $R_r^{\mathbf{t}}(n, n) < \mathrm{tw}_{r-1}(n^{O(1)})$, where $\mathrm{tw}_k(x)$ is a tower of 2's of height k with an x on the top. This bound is also the best possible if $\min\{d, D, m\}$ is sufficiently large with respect to r. They conjectured that in the asymmetric case, we have $R_3^{\mathbf{t}}(s, n) < n^{O(1)}$ for fixed s. We refute this conjecture by showing that $R_3^{\mathbf{t}}(4, n) > n^{(\log n)^{1/3 - o(1)}}$ for some complexity $\mathbf{t}$.

In addition, motivated by the results of Bukh-Matoušek and Basit-Chernikov-Starchenko-Tao-Tran, we study the complexity of the Ramsey problem when the defining polynomials are linear, that is, when $D = 1$. In particular, we prove that $R_r^{d,1,m}(n, n) \leq 2^{O(n^{4r^2 m^2})}$, while from below, we establish $R_r^{1,1,1}(n, n) \geq 2^{\Omega(n^{\lfloor r/2 \rfloor - 1})}$.

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1   Introduction

Given positive integers r, s, n, the *Ramsey number* $R_r(s, n)$ denotes the smallest N such that every r-uniform hypergraph on N vertices contains either a clique of size s, or an independent set of size n. For convenience, we write $R_r(n)$ instead of $R_r(n, n)$. In the case of graphs, that is $r = 2$, classical results of Erdős and Szekeres [15] and Erdős [12] tell us that $R_2(n) = 2^{\Theta(n)}$, and in case s is fixed and n is sufficiently large, we have $R_2(s, n) = n^{\Theta(s)}$. However, in case $r \geq 3$, the Ramsey numbers are less understood. Erdős and Rado [11] and Erdős, Hajnal and Rado [14] show that

$$\mathrm{tw}_{r-1}(\Omega(n^2)) < R_r(n) < \mathrm{tw}_r(O(n)).$$

Also, in the asymmetric case, we have $R_r(s, n) = \mathrm{tw}_{r-1}(n^{\Theta_{r,s}(1)})$ for $s \geq r + 2$ [14, 18], and $R_3(4, n) = 2^{n^{\Theta(1)}}$ [10]. Here, $\mathrm{tw}_k(x)$ is the *tower function* defined as $\mathrm{tw}_1(x) := x$ and $\mathrm{tw}_k(x) := 2^{\mathrm{tw}_{k-1}(x)}$. Hence, there is an almost exponential gap between the lower and upper bound for $R_r(n)$ in case $r \geq 3$, and it is a major open problem to close this gap. Note that, however, the rough order of the asymmetric Ramsey number $R_r(s, n)$ is more understood, at least up to the height of the required tower. See [7] for recent developments.

Yet the situation changes if we restrict our attention to hypergraphs that arise from geometric considerations. To this end, an r-uniform hypergraph H is *semi-algebraic of complexity* (d, D, m) if the following holds. There is an enumeration $v_1, \ldots, v_N$ of the vertices of H, an assignment $v_i \mapsto p_i$ with $p_i \in \mathbb{R}^d$ for $i \in [N]$, and m polynomials $f_1, \ldots, f_m : (\mathbb{R}^d)^r \mapsto \mathbb{R}$ of (total) degree at most D such that for $1 \leq i_1 < \cdots < i_r \leq N$, whether $\{v_{i_1}, \ldots, v_{i_r}\}$ is an edge of H depends only on the sign-pattern of $(f_1(p_{i_1}, \ldots, p_{i_r}), \ldots, f_m(p_{i_1}, \ldots, p_{i_r}))$. More precisely, there is a function $\Phi : \{+, -, 0\}^m \mapsto \{\mathrm{True}, \mathrm{False}\}$ such that $\{v_{i_1}, \ldots, v_{i_r}\}$ is an edge if and only if

$$\Phi(\mathrm{sign}(f_1(p_{i_1}, \ldots, p_{i_r})), \ldots, \mathrm{sign}(f_m(p_{i_1}, \ldots, p_{i_r}))) = \mathrm{True}.$$

Semi-algebraic graphs and hypergraphs of bounded complexity provide a general model to study certain geometric structures, such as intersection and incidence graphs of geometric objects, order types of point configurations, convex subsets of the plane, and so on. The semi-algebraic Ramsey number $R_r^{\mathbf{t}}(s, n)$ denotes the smallest N such that any r-uniform semi-algebraic hypergraph of complexity $\mathbf{t}$ on N vertices contains either a clique of size s or an independent set of size n. Alon, Pach, Pinchasi, Radoičić and Sharir [1] proved that $R_2^{\mathbf{t}}(n) = n^{\Theta(1)}$, which was extended by Conlon, Fox, Pach, Sudakov and Suk [6] to $R_r^{\mathbf{t}}(n) = \mathrm{tw}_{r-1}(n^{O(1)})$ for general r. In [6] and [9], matching lower bounds are provided in case the parameters d, D, m are sufficiently large with respect to r. Specifically, for every $r \geq 2$, there exists $\mathbf{t}$ such that $R_r^{\mathbf{t}}(n) = \mathrm{tw}_{r-1}(n^{\Theta(1)})$. Here and later, the constants hidden by the $O(.), \Omega(.), \Theta(.)$ notation might depend on $r, \mathbf{t}$ and s, unless specified otherwise.

1.1   Asymmetric Ramsey numbers

In contrast, asymmetric semi-algebraic Ramsey numbers appear to be more mysterious in case $r \geq 3$. For uniformity $r = 3$, in the special subcase $d = 1$, it was established in [6]

that $R_3^{\mathbf{t}}(s,n) < 2^{(\log n)^{O(1)}}$. Furthermore, if $d \geq 2$, a result of Suk [19] shows that

$$R_3^{\mathbf{t}}(s,n) < 2^{2^{(\log n)^{1/2+o(1)}}} = 2^{n^{o(1)}}.$$

However, the best known lower bound constructions provide only polynomial growth, which leads to the natural conjecture that $R_3^{\mathbf{t}}(s,n) = n^{O(1)}$, formulated in both [6] and [19]. Our first main result refutes this conjecture.

Theorem 1.1. *There exists* $\mathbf{t} = (d, D, m)$ *such that*

$$R_3^{\mathbf{t}}(4,n) > n^{(\log n)^{1/3-o(1)}}.$$

1.2 Semi-linear hypergraphs

As discussed above, if d, D, m are sufficiently large with respect to r, then $R_r^{d,D,m}(n) = \text{tw}_{r-1}(n^{\Omega(1)})$. In the constructions provided by both [6] and [9], the parameters d and D grow with r. In particular, [9] shows that one can take $d = r - 3$ for $r \geq 4$. Furthermore, the Veronese mapping[1] implies that every r-uniform semi-algebraic hypergraph of complexity (d, D, m) is also of complexity (d', r, m) for some d' depending only on d and D. However, this raises the question whether the upper bound $R_r^{d,D,m}(n) < \text{tw}_{r-1}(n^{O(1)})$ can be significantly improved if we assume that d or D are small compared to r. In support of this, Bukh and Matoušek [4] showed that if $d = 1$, that is, when the vertices of the hypergraph correspond to points on the real line, then any r-uniform semi-algebraic hypergraph of complexity $(1, D, m)$ containing no clique or independent set of size n has at most $2^{2^{O(n)}}$ vertices (in [4], the constant hidden by the $O(.)$ notation might depend on the defining polynomials, but a careful inspection of their proof yields that it can be bounded only by a function of D, m and r as well). Also, this bound is the best possible if D and m are sufficiently large. In this paper, we consider what happens if we bound the parameter D instead, that is, the degrees of the defining polynomials.

A semi-algebraic hypergraph of complexity (d, D, m) is *semi-linear*, if $D = 1$, that is, all defining polynomials are linear functions[2] The study of semi-linear hypergraphs was initiated by Basit, Chernikov, Starchenko, Tao and Tran [3], who considered these hypergraphs in the setting of Zarankiewicz's problem. There are many extensively studied families of graphs that are semi-linear of bounded complexity, for example intersection graphs of axis-parallel boxes in $\mathbb{R}^d$, circle graphs, and shift graphs. Motivated by the large literature (e.g. [2, 5, 8, 13, 17]) concerned with the Ramsey properties of such families, Tomon [22] studied the Ramsey properties of semi-linear graphs and showed that $R_2^{d,1,m}(s,n) \leq n^{1+o(1)}$ holds for every fixed s, d and m. This already shows a behavior unique to semi-linearity, as a construction of Suk and Tomon [20] shows that $R_2^{d,2,m}(3,n) = \Omega(n^{4/3})$

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[1]A Veronese mapping sends $(x_1, \ldots, x_d) \in \mathbb{R}^d$ to some point whose coordinates are monomials of $x_1, \ldots, x_d$. E.g. $(x_1, x_2, x_3) \mapsto (x_1^2 x_2, x_2^2 x_3^2, x_1 x_2 x_3, x_3^3)$.

[2]To clarify, e.g. $(x_1, x_2) \mapsto 2x_1 + 3x_2 + 5$ is a linear function, but $(x_1, x_2) \mapsto x_1 x_2 + 3x_1 + 3$ is *not* linear, it is only multi-linear.

for some d and m. Tomon [22] also proposed the problem of determining the Ramsey numbers of r-uniform semi-linear hypergraphs for $r \geq 3$. Our second main result settles this problem.

Theorem 1.2. *For every triple of positive integers r, d, m, there exists $c = c(r, m) > 0$ such that*
$$R_r^{d,1,m}(n) \leq 2^{cn^{4r^2m^2}}.$$

Let us highlight that the bound in Theorem 1.2 does not depend on the dimension d, only on the uniformity r and the number of polynomials m. From below, in case $r \geq 3$, the semi-linear Ramsey number grows at least exponentially, showing that Theorem 1.2 is sharp up to the value of c and the exponent $4r^2m^2$. Indeed, let H be the 3-uniform hypergraph on vertex set $\{1, \ldots, N\}$ in which for $x < y < z$, $\{x, y, z\}$ is an edge if $x + z < 2y$. Then H is semi-linear of complexity $(1, 1, 1)$, and it is easy to show that $\omega(H), \alpha(H) \leq \lceil \log_2 N \rceil + 1$. Thus, $R_3^{1,1,1}(n) \geq 2^{\Omega(n)}$. We show that even faster growth can be achieved by examining certain more convoluted constructions of higher uniformity.

Theorem 1.3. *For every $r \geq 4$, there exists a constant $c > 0$ such that*
$$R_r^{1,1,1}(n) \geq 2^{cn^{\lfloor r/2 \rfloor - 1}}.$$

2 A lower bound for $R_3^{\mathsf{t}}(4, n)$

In this section, we outline the construction for Theorem 1.1, which builds on a variant of the famous stepping-up lemma of Erdős and Hajnal (see [16]).

Given distinct $\alpha, \beta \in \{0, 1\}^N$, let $\delta(\alpha, \beta) := \min\{i : \alpha(i) \neq \beta(i)\}$. Let $\prec$ be the lexicographical order over $\{0, 1\}^N$, i.e. $\alpha \prec \beta \Leftrightarrow \alpha(\delta(\alpha, \beta)) < \beta(\delta(\alpha, \beta))$. An important property of $\delta(\cdot, \cdot)$ is that for any $\alpha_1 \prec \cdots \prec \alpha_\ell$, there is a unique i which achieves the minimum of $\delta(\alpha_i, \alpha_{i+1})$.

Now we define our notion of the step-up.

Definition 1. The *step-up* of a graph G is the 3-uniform hypergraph H on vertex set $\{0, 1\}^N$ defined as follows. For $\alpha, \beta, \gamma \in \{0, 1\}^N$ with $\alpha \prec \beta \prec \gamma$, we have $\{\alpha, \beta, \gamma\} \in E(H)$ if and only if $\delta(\alpha, \beta) < \delta(\beta, \gamma)$ and $\{\delta(\alpha, \beta), \delta(\beta, \gamma)\} \in E(G)$.

The next lemma relates the clique and independence numbers of both graphs.

Lemma 2.1. $\omega(H) \leq \omega(G) + 1$ *and* $\alpha(H) \leq N^{\alpha(G)} + 1$.

By a construction of Suk and Tomon [20], there exists a semi-algebraic graph G on $\Theta(m^{4/3})$ vertices with $\omega(G) = 2$ and $\alpha(G) \leq 2m$ for all $m \in \mathbb{N}$. The methods in [6, 9] show that the step-up of G, denoted by H, remains semi-algebraic. Pick m such that $n = \Theta((m^{4/3})^{2m})$, i.e. $m = \Omega(\log n / \log \log n)$. Then, $|V(H)| = 2^{\Theta(m^{4/3})} = n^{\Omega((\log n)^{1/3 - o(1)})}$, $\omega(H) = 3$, $\alpha(H) \leq |V(G)|^{2m} + 1 < n$. This finishes the proof.

3   Semi-linear hypergraphs

In this section, we outline the proof of Theorem 1.2, i.e. $R_r^{d,1,m}(n) \leq 2^{O(n^{4r^2m^2})}$. Let H be an r-uniform semi-linear hypergraph on vertex set $[N]$ of complexity $(d, 1, m)$. We observe that H is the Boolean combination of $2m$ semi-linear hypergraphs $H_1, \ldots, H_{2m}$, where H_i is defined by a matrix $P_i \in \mathbb{R}^{r \times N}$ as follows: for $1 \leq q_1 < \cdots < q_r \leq N$, we have $\{q_1, \ldots, q_r\} \in H$ if and only if $\sum_{i=1}^{r} P(i, q_i) < 0$. Therefore, our goal is to find $C \subseteq [N]$ of size $(\log N)^{\Omega_{r,m}(1)}$ such that for each $i \in [2m]$, C is either a clique or an independent set in H_i. We will find such a C by trimming and transforming our matrices in several steps.

For $\delta \in \mathbb{R}$, the *shift of a sequence* $x_1, \ldots, x_N$ *by* δ is the sequence $x_1 + \delta, \ldots, x_N + \delta$. Given $\Delta > 1$ and $\tau \in \{-, +\} \times \{\searrow, \nearrow\}$, a sequence $x_1, \ldots, x_N$ is called (Δ, τ)-*exponential* if for all $i \in [N-1]$, we have $0 < x_i < x_{i+1}/\Delta$ in the case $\tau = (+, \nearrow)$; $0 < x_{i+1} < x_i/\Delta$ in the case $\tau = (+, \searrow)$; $0 < (-x_i) < (-x_{i+1}/\Delta)$ in the case $\tau = (-, \nearrow)$; $0 < (-x_{i+1}) < (-x_i/\Delta)$ in the case $\tau = (-, \searrow)$. Also, say that a sequence is Δ-*exponential* if it is (Δ, τ)-exponential for some $\tau \in \{-, +\} \times \{\searrow, \nearrow\}$.

Lemma 3.1. *For every q there exists $c = c(q) > 0$ such that the following holds. Let $M \in \mathbb{R}^{q \times N}$ be a matrix such that no row contains repeated elements. Then M contains a $q \times N'$ sized submatrix for $N' = c(\log N)^{1/q}$ such that every row of M is $2q$-exponential.*

Applying Lemma 3.1 to the concatenation of the matrices P_ℓ for $\ell \in [2m]$, we find a subset $I \subset [N]$ of size $N' = c(\log N)^{1/2mr}$ (say $I' = [N']$) such that for each ℓ, the submatrix P'_ℓ of P_ℓ induced by columns in I is $2r$-exponential. Let H' be the subgraph of H induced by I, and H'_i be the subgraph of H'_i induced by I. It is easy to see that every H'_ℓ is defined by P'_ℓ, and that H' is a Boolean combination of $H'_1, \ldots, H'_{2m}$. Theorem 1.2 is then an easy consequence of the following key lemma, which guarantees $C \subset I$ of size $(N')^{\Omega_{r,m}(1)}$ such that for all $i \in [2m]$, C is either a clique or an independent set in H'_i.

Lemma 3.2. *For every r and k, there exists $c = c(r, k) > 0$ such that the following holds. Let $P_1, \ldots, P_k$ be $r \times N$ matrices where all rows are $2r$-exponential. Then there exists $C \subset [N]$ such that $|C| \geq cN^{\frac{1}{rk-k+1}}$ and C is a clique or an independent set in H_i for every $i \in [k]$.*

The proof of this lemma builds on the following observation. Using that each row of P_ℓ is $(2r)$-exponential, whether $\{q_1, \ldots, q_r\}$ is an edge of H_ℓ depends (essentially) on the maximum of $H_\ell(1, q_1), \ldots, H_\ell(r, q_r)$. Further details are omitted.

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