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How connectivity affects the extremal number of trees

(Extended abstract)

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Abstract

The Erdős-Sós conjecture states that the maximum number of edges in an n-vertex graph without a given k-vertex tree is at most $\frac{n(k-2)}{2}$. Despite significant interest, the conjecture remains unsolved. Recently, Caro, Patkós, and Tuza considered this problem for host graphs that are connected. Settling a problem posed by them, for a k-vertex tree T, we construct n-vertex connected graphs that are T-free with at least $(1/4 - o_k(1))nk$ edges, showing that the additional connectivity condition can reduce the maximum size by at most a factor of 2. Furthermore, we show that this is optimal: there is a family of k-vertex brooms T such that the maximum size of an n-vertex connected T-free graph is at most $(1/4 + o_k(1))nk$.

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1 Introduction

In extremal graph theory, a central focus is determining the extremal number of various graphs. The extremal number, denoted by $\mathrm{ex}(n, F)$, is the maximum number of edges in an n-vertex graph that does not contain a graph F as a subgraph, not necessarily induced. While the asymptotic behavior of this function has been determined for all non-bipartite

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graphs by Erdős, Stone, and Simonovits [6, 10], the behavior for bipartite graphs remains open with significant interest from the community.

The Erdős-Gallai theorem [9], established in 1959, studied the k-vertex path P_k, stating that $\mathrm{ex}(n, P_k) \leqslant \frac{n(k-2)}{2}$, and the maximum value is achieved by $\cup K_{k-1}$, the disjoint union of K_{k-1} when $k-1 | n$. Once the extremal number of the path is determined, extending the question to a tree is a natural next step. In 1962, Erdős and Sós [7] conjectured that for any k-vertex tree, its extremal number is at most $\frac{n(k-2)}{2}$, and again the disjoint union of K_{k-1} serves as an example for tightness.

Motivated by the fact that the conjectured maximizer $\cup K_{k-1}$ is not connected, a natural variant is to consider host graphs that are connected, see e.g [15, 4]. Formally, the connected extremal number $\mathrm{ex}_c(n, F)$ is the maximum number of edges in an n-vertex connected graph without a subgraph isomorphic to F. While the additional connectivity condition does not affect the asymptotics of the extremal number when the forbidden graph is non-bipartite or 2-edge-connected, Caro, Patkós, and Tuza [4] investigated what effect it has for trees. Notice that, in contrast with the classical extremal number, its connected relative is not even a monotone function of n. Indeed, for paths, it is known that for every $k \geqslant 10$, $\mathrm{ex}_c(k, P_k) = \binom{k-2}{2} + 2 < \binom{k-1}{2} = \mathrm{ex}_c(k-1, P_k)$.

Such connected variant for trees was in fact studied before and could date back to the work of Kopylov [19] in 1977, in which he resolved the problem for paths, showing that for $n \geqslant k \geqslant 4$,

$$\mathrm{ex}_c(n, P_k) = \max\left\{\binom{k-2}{2} + (n-(k-2)), \left\lfloor\frac{k-2}{2}\right\rfloor (n - \left\lceil\frac{k}{2}\right\rceil) + \binom{\lceil\frac{k}{2}\rceil}{2}\right\}. \qquad (1)$$

Later, Balister, Győri, Lehel, Schelp [1] characterized extremal graphs for every n. There are also recent developments towards the stability version of this theorem by Füredi, Kostochka, Luo, Verstraëte [13, 14].

By Erdős-Gallai theorem and Kopylov's result (1), we see that the asymptotic of the maximum number of edges in an n-vertex P_k-free graph does not change by an additional connectivity constraint as $n \to \infty$ and $k \to \infty$. Caro, Patkós, and Tuza [4] studied how much smaller $\mathrm{ex}_c(n, T)$ can be for a k-vertex tree T, compared to $\frac{n(k-2)}{2}$. Formally, they defined

$$\gamma_k := \inf\left\{\limsup_{n\to\infty} \frac{\mathrm{ex}_c(n, T_k)}{\frac{n(k-2)}{2}} : T_k \text{ is a } k\text{-vertex tree}\right\} \text{ and } \gamma := \lim_{k\to\infty} \gamma_k. \qquad (2)$$

It is not hard to see that this limit exists. From above, Caro, Patkós, and Tuza [4] found a family of trees whose connected extremal number is asymptotically smaller, yielding $\gamma \leqslant \frac{2}{3}$. From below, for every tree, they gave constructions showing that $\gamma \geqslant \frac{1}{3}$. They asked where the truth lies between $\frac{1}{3}$ and $\frac{2}{3}$.

Our main result settles this problem.

Theorem 1.1. *Let γ be as defined in (2), we have $\gamma = \frac{1}{2}$.*

To obtain the lower bound $\gamma \geqslant \frac{1}{2}$, we provide several families of different constructions depending on the 'center of mass' of the forbidden tree (see Section 2.1), realizing the following.

Theorem 1.2. *For any k-vertex tree T_k, we have*

$$\mathrm{ex}_c(n, T_k) \geqslant \left(\frac{1}{4} - o_k(1)\right) kn.$$

On the other hand, we determine the exact connected extremal number of brooms with k vertices and diameter d, denoted by $B(k, d)$, for large enough n. In particular, $B(k, d)$ is the graph obtained from a path of $d + 1$ vertices by blowing up a leaf to an independent set of size $k - d$. The following theorem is stated using graphs $\mathsf{G}_{n,\cdot,\cdot}$ and $\mathsf{F}_{n,\cdot,\cdot}$, which are defined in Sections 2.2 and 3.1. Some of these graphs (so-called edge blow-up of stars) have been studied before, see e.g. [5, 8, 27]. As the path is also a broom, the result below can be viewed as an extension of Kopylov's result (1).

Theorem 1.3. *For every integer k and d such that $k \geqslant d + 2 \geqslant 8$, and $n \geqslant k^{dk}$ we have*

$$\mathrm{ex}_c(n, B(k, d)) = \begin{cases} e(\mathsf{G}_{n,d,\lfloor \frac{d-1}{2} \rfloor}) & \text{if } d \geqslant \frac{k+5}{2}, \\ \max\{e(\mathsf{G}_{n,d,\lfloor \frac{d-1}{2} \rfloor}), e(\mathsf{F}_{n,\frac{k+2}{2},1})\} & \text{if } d = \frac{k+2}{2} \text{ or } \frac{k+4}{2}, \\ \max\{e(\mathsf{G}_{n,d,\lfloor \frac{d-1}{2} \rfloor}), e(\mathsf{F}_{n,d,2}), e(\mathsf{F}_{n,d,3})\} & \text{if } d = \frac{k+3}{2}, \\ \left\lfloor \frac{(k-d)n}{2} \right\rfloor & \text{if } d \leqslant \frac{k+1}{2}. \end{cases}$$

As a corollary, we get the matching upper bound $\gamma \leqslant \frac{1}{2}$ as $\mathrm{ex}_c(n, B(k, \lceil \frac{k}{2} \rceil)) = \left(\frac{1}{4} + o_k(1)\right) kn$.

2 Overview of the proof of Theorem 1.2

In this section, we first introduce a key concept: the barycenter of a tree. We then provide three special graphs $\mathsf{G}_{n,\frac{k-c}{2},\frac{k-c}{4}}$, $\mathsf{S}_{n,x}$ and $\mathsf{P}_{n,x}$, each of which has $(\frac{1}{4} - o_k(1))kn$ edges, where c is a constant. By considering the degree of the barycenter vertex of the tree T_k, we can find a T_k-free graph G such that G is isomorphic to one of $\mathsf{G}_{n,\frac{k-c}{2},\frac{k-c}{4}}$, $\mathsf{S}_{n,x}$ and $\mathsf{P}_{n,x}$ for some x.

2.1 The Barycenter of a tree

For any tree T on k vertices, we call a vertex v of T *a barycenter* if v belongs to a largest connected component of $T - e$ for every edge e in T, that is, the vertex v belongs to the component of size at least $\lceil \frac{k}{2} \rceil$ in the graph obtained from T by removing an edge e.

Proposition 2.1. *Every tree has either a unique barycenter, or there are exactly two barycenters in the tree joined by an edge.*

2.2 Constructions of various classes of graphs

The family $\mathsf{G}_{n,\cdot,\cdot}$. We first define the family of extremal graphs for (1). Recall '$\cup$' denotes the disjoint union of graphs, '+' denotes the join of the graphs, and $\overline{K}_t$ denotes an independent set of size t. For $n \geqslant k \geqslant 2s$ let $\mathsf{G}_{n,k,s} := \left(K_{k-2s} \cup \overline{K}_{n-k+s}\right) + K_s$, see Figure 2.1. Note that for every n and k, there exists a constant a such that $k < a < 2k$ and the only extremal graphs achieving equality in (1) are $\mathsf{G}_{n,k-1,1}$ for $n \leqslant a$ and $\mathsf{G}_{n,k-1,\lfloor\frac{k-2}{2}\rfloor}$ for $n \geqslant a$. Clearly, $e(\mathsf{G}_{n,\frac{k-c}{2},\frac{k-c}{4}}) = (\frac{1}{4} - o_k(1))kn$.

The families $\mathsf{S}_{n,x}$ and $\mathsf{P}_{n,x}$. Let x be an integer such that $\frac{k}{2} < x < k$ or $x = \left\lfloor\frac{k-2}{2}\right\rfloor$. For the sake of simplicity of the write-up, we denote

$$a_x := \begin{cases} \left\lfloor \frac{2x^2}{k} \right\rfloor - 2 & \text{if } \frac{k}{2} < x < k, \text{ and} \\ x & \text{if } x = \left\lfloor\frac{k-2}{2}\right\rfloor. \end{cases}$$

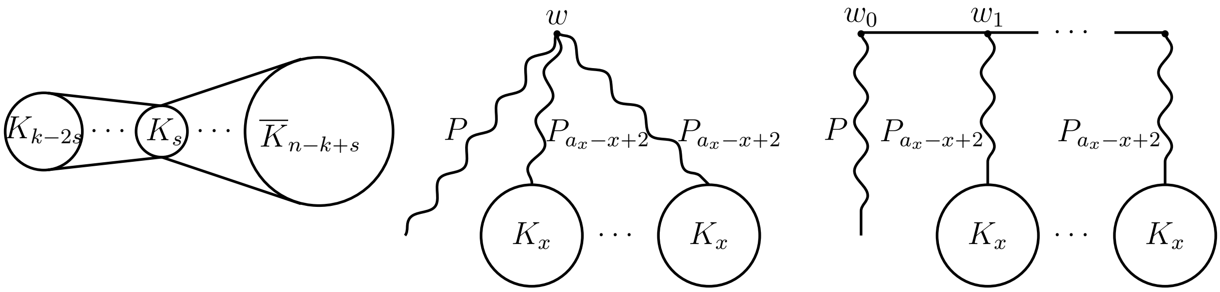

Figure 2.1: The graph $\mathsf{G}_{n,k,s}$ on the left, the graph $\mathsf{S}_{n,x}$ in the middle and the graph $\mathsf{P}_{n,x}$ on the right.

Let $\mathsf{S}_{n,x}$ be a graph consisting of $\left\lfloor\frac{n-1}{a_x}\right\rfloor$ vertex disjoint K_x with pendant paths of length $a_x - x$, a path of $n - 1 - a_x\left\lfloor\frac{n-1}{a_x}\right\rfloor$ vertices and a vertex w adjacent with an endpoint of each of these paths.

Let $\mathsf{P}_{n,x}$ be a graph consisting of $\left\lfloor\frac{n}{a_x+1}\right\rfloor$ vertex disjoint K_x with pendant paths of length $a_x - x + 1$ with the terminal leaf w_i ($i \geqslant 1$), a path of $n - (a_x+1)\left\lfloor\frac{n}{a_x+1}\right\rfloor$ vertices with a terminal leaf w_0, such that $w_0 w_1 \dots w_{\lfloor\frac{n}{a_x+1}\rfloor}$ is a path.

It is easy to see that $e(\mathsf{S}_{n,x}) = (\frac{1}{4} - o_k(1))kn$ and $e(\mathsf{P}_{n,x}) = (\frac{1}{4} - o_k(1))kn$.

Now we give the overview of the proof of Theorem 1.2: Let v be a barycenter of the tree T_k, which exists by Proposition 2.1. Let $x_1 = \left\lfloor\frac{k-2}{2}\right\rfloor$ and $x_2 = \left\lfloor\frac{k}{\sqrt{2}}\right\rfloor$. We split the proof into three cases: $d(v) = 2$, $d(v) \geqslant 4$ and $d(v) = 3$. For the cases $d(v) = 2$ and $d(v) \geqslant 4$, we show that S_{n,x_1} and P_{n,x_1} are T_k-free, respectively. For the case $d(v) = 3$, we show that either T_k can not be embedded in S_{n,x_2} or P_{n,x_2}, or T_k contains two vertex disjoint sub-trees

S_1 and S_2, each of which is isomorphic to a spider[1] with the central vertex of degree at most three and $v(S_1) + v(S_2) \geqslant \frac{k-c}{2}$ for some constant c and thus T_k can not embedding in $\mathsf{G}_{n,2(\lfloor \frac{k-c}{4} \rfloor -5), \lfloor \frac{k-c}{4} \rfloor -5}$. In these three cases, we can always get the desired lower bound.

3 Overview of the proof of Theorem 1.3

In this section, we first construct some edge blow-up of stars $\mathsf{F}_{n,d,\cdot}$ in Section 3.1, which $e(\mathsf{F}_{n,d,\cdot})$ achieves the lower bound of $\mathrm{ex}_c(n, B(k,d))$. To prove the upper bound of $ex_c(n, B(k,d))$ we need some theorems, see Section 3.2.

3.1 The family $\mathsf{F}_{n,\cdot,\cdot}$

For $n > d \geqslant 2$, let $\mathsf{F}_{n,d,1}$ be the n-vertex connected graph such that every maximal 2-connected block is a clique of size $d-1$ except at most one clique of size $n - \lfloor \frac{n-1}{d-2} \rfloor (d-2)$, all sharing a common vertex. Thus if $n - 1 = (d-2)p_1 + q_1$ for non-negative integers p_1 and q_1 such that $0 \leqslant q_1 < d-2$ then

$$e(\mathsf{F}_{n,d,1}) = p_1 \binom{d-1}{2} + \binom{q_1+1}{2} = \frac{(d-1)(n-1)}{2} - \frac{q_1(d-2-q_1)}{2}.$$

Let $\mathsf{F}_{n,d,2}$ be the n-vertex connected graph such that every maximal 2-connected block is a clique with one of size $d-1$, the rest of size $d-2$ except at most one clique of size $n - 1 - \lfloor \frac{n-2}{d-3} \rfloor (d-3)$, all sharing a common vertex. Thus if $n - 2 = (d-3)p_2 + q_2$ for integers p_2 and q_2 such that $p_2 \geqslant 1$ and $0 \leqslant q_2 < d-3$ then

$$e(\mathsf{F}_{n,d,2}) = p_2 \binom{d-2}{2} + d - 2 + \binom{q_2+1}{2} = \frac{(d-2)n}{2} - \frac{q_2(d-3-q_2)}{2}.$$

For an even integer d, let $n - 1 = (d-3)p_3 + q_3$ for integers p_3 and q_3 such that $0 \leqslant q_3 < d-3$. If $p_3 \geqslant q_3$, let $\mathsf{F}_{n,d,3}$ be the n-vertex connected graph such that it contains p_3 maximal 2-connected blocks $G_1, \ldots, G_{p_3}$ all sharing a common vertex v with $p_3 - q_3$ of them being the cliques of size $d-2$. The rest of the maximal 2-connected blocks G_i are the cliques of size $d-1$ without a perfect matching of $G_i - v$. We have

$$e(\mathsf{F}_{n,d,3}) = \frac{(d-2)(n-1)}{2}.$$

If $p_3 < q_3$, let $\mathsf{F}_{n,d,3}$ be the n-vertex connected graph such that it contains $p_3 + 1$ maximal 2-connected blocks $G_1, \ldots, G_{p_3+1}$ all sharing a common vertex v with p_3 of them being the cliques of size $d-1$ without a perfect matching of $G_i - v$ and the remaining one is a clique of size $q_3 - p_3 + 1$. We have

$$e(\mathsf{F}_{n,d,3}) = p_3 \frac{(d-2)^2}{2} + \binom{q_3 - p_3 + 1}{2} = \frac{(d-2)(n-1)}{2} - \frac{(q_3-p_3)(d-3-(q_3-p_3))}{2}.$$

[1]Spider is a tree with all vertices of degree at most two, except one vertex of any degree, referred to as the central vertex of the spider.

3.2 Other tools

Let $\mathcal{C}_{\geqslant k}$ denote the class of cycles of length at least k. For a class of graphs $\mathcal{F}$, the Turán number of $\mathcal{F}$ is the maximum number of edges in a graph not containing a subgraph F for all $F \in \mathcal{F}$, denoted by $\mathrm{ex}(n, \mathcal{F})$.

Woodall [25] and independently Kopylov [19] improved Erdős-Gallai theorem [9] for long cycles and obtained the following exact result for every n, see also [12].

Theorem 3.1 (Woodall [25], Kopylov [19]). *Let $n = p(k-2) + q + 1$, where $0 \leqslant q < k-2$ and $k \geqslant 3$, $p \geqslant 1$,*

$$\mathrm{ex}(n, \mathcal{C}_{\geqslant k}) = p\binom{k-1}{2} + \binom{q+1}{2} = \frac{(k-1)(n-1)}{2} - \frac{q(k-2-q)}{2}.$$

Theorem 3.2 (Kopylov [19], Woodall [25], Fan, Lv and Wang [11]). *Suppose $n \geqslant k \geqslant 5$, then every 2-connected n-vertex $\mathcal{C}_{\geqslant k}$-free graph contains at most*

$$\max\left\{\binom{k-2}{2} + 2(n-(k-2)), \left\lfloor \frac{k-1}{2} \right\rfloor \left(n - \left\lceil \frac{k+1}{2} \right\rceil\right) + \binom{\lceil \frac{k+1}{2} \rceil}{2}\right\} \text{ edges.}$$

The extremal graphs are $\mathsf{G}_{n,k,2}$ and $\mathsf{G}_{n,k,\lfloor \frac{k-1}{2} \rfloor}$.

Now we give a overview of the proof of Theorem 1.3: For the lower bound of $\mathrm{ex}_c(n, B(k,d))$, we consider the following $B(k,d)$-free graphs: $\mathsf{G}_{n,d,\lfloor \frac{d-1}{2} \rfloor}$, an almost $(k-d)$-regular graph, $\mathsf{F}_{n,\frac{k+2}{2},1}$ if $d = \frac{k+2}{2}$ or $\frac{k+4}{2}$, $\mathsf{F}_{n,d,2}$ and $\mathsf{F}_{n,d,3}$ if $d = \frac{k+3}{2}$.

For the matching upper bound, let G be an n-vertex connected $B(k,d)$-free graph with $n \geqslant k^{dk}$ and let v be a vertex of G such that $d(v) = \Delta(G)$. We divide the proof into three cases depending on the value of $\Delta(G)$. If $\Delta(G) \leqslant k-d$ then we easily get the desired upper bound $\left\lfloor \frac{(k-d)n}{2} \right\rfloor$. If $k-d+1 \leqslant \Delta(G) \leqslant k-2$, then there exists a path of at least dk vertices starting at v by considering a breadth-first search tree from the vertex v, and then by pigeonhole principle we will find a copy of $B(k,d)$ in G resulting in a contradiction. For the last case $\Delta(G) \geqslant k-1$, if G is 2-connected then we get the desired upper bound by Theorem 3.2; otherwise, we assume G is not 2-connected and its maximal 2-connected blocks are $G_1, G_2, \ldots, G_s$. By Theorem 3.1, the circumference of G is either $d-2$ or $d-1$ or we have the desired upper bound. And by Theorem 3.2, each $e(G_i)$ is bounded for $i \in [s]$. By using the properties of G (the circumference and $B(k,d)$-free), we can get the desired upper bound.

4 Concluding remarks

For the Turán problem, we determine the maximum effect the additional connectivity condition could have over all trees. An interesting future direction of research would be to identify appropriate parameters (if exist) of a tree that determine the asymptotic behavior of its connected extremal number. The constructions in Sections 2 and 3 could be useful for this problem.

The reader interested in the problems and extensions related to Erdős-Sós conjecture we refer to the following papers [2, 22, 17, 20, 21, 23, 26, 3], extensions for Berge hypergraphs see [16, 18], extensions for colored graphs see [24].

Caro, Patkós, Tuza [4] asked whether the connected extremal number becomes monotone eventually. In particular, for every graph F, there exists a constant N_F, such that for every $n \geqslant N_F$, $\mathrm{ex}_c(n, F) \leqslant \mathrm{ex}_c(n+1, F)$. We observe that this is true when F contains a cycle.

Proposition 4.1. *For every graph F containing a cycle, there exists a constant N_F such that for every $n > N_F$ we have $\mathrm{ex}_c(n, F) < \mathrm{ex}_c(n+1, F)$.*

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