

COPS AND ROBBER ON HYPERBOLIC MANIFOLDS

(EXTENDED ABSTRACT)

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Abstract

The Cops and Robber game on geodesic spaces is a pursuit-evasion game with discrete steps which captures the behavior of the game played on graphs, as well as that of continuous pursuit-evasion games. One of the outstanding open problems about the game on graphs is to determine which graphs embeddable in a surface of genus g have largest cop number. It is known that the cop number of genus g graphs is $O(g)$ and that there are examples whose cop number is $\tilde{\Omega}(\sqrt{g})$. The same phenomenon occurs when the game is played on geodesic surfaces.

In this paper we obtain a surprising result when the game is played on a surface with constant curvature. It is shown that two cops have a strategy to come arbitrarily close to the robber, independently of the genus. For special hyperbolic surfaces we also give upper bounds on the number of cops needed to catch the robber. Our results generalize to higher-dimensional hyperbolic manifolds.

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1 Introduction

The Cops and Robber game is a pursuit-evasion game. The game is commonly played on graphs [1, 5, 6, 7, 9, 12, 16, 19], and as a new variant on geodesic spaces [15, 21, 22]. The

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players in the Cops and Robber game are the robber r and k cops c_1, \dots, c_k . On graphs the players occupy vertices while on geodesic spaces the players occupy points in space. The game is played in rounds. The robber chooses initial positions for the players r^0, c_1^0, \dots, c_k^0 and in each round of the game the players can move to a new position¹. Each round of the game has two turns, the first one for the robber and the second one for the cops. In particular, each of the cops can make a step at the cops' turn. When the game is played on a graph, the players are allowed to move to an adjacent vertex at their turn. When the game is played on a geodesic space X , the robber chooses an agility function $\tau : \mathbb{N} \rightarrow \mathbb{R}_+$ at the beginning of the game such that $\sum_{n \geq 1} \tau(n) = \infty$. In the n -th round, each player makes a step of length at most $\tau(n)$ (at their turn). The position of the robber r after round n is denoted as r^n , and the position of the cop c_i as c_i^n . In the following we give the robber the pronoun he, him, while the cops have the pronoun she, her. We say the cops *catch* the robber if at some point in the game the cop c_i occupies the same position as the robber r . If the robber is not caught, we say the robber *escapes*. The *cop number* $c(G)$ of a graph G is the minimum number of cops that can catch the robber (regardless of the robber's strategy and initial positions). For a geodesic space X we denote by $c_0(X)$ the *cop catch number*, which is the minimum number of cops that can catch the robber. Further, if the game is played on a geodesic space, we say that *the cops win* the game if

$$\inf_{n,i} d(c_i^n, r^n) = 0. \quad (1)$$

If the cops do not win the game, we say that *the robber wins* the game, which means that he can stay at distance at least ε away from the cops, for some $\varepsilon > 0$. For a geodesic space we denote by $c(X)$ the *cop win number*, that is the minimum number of cops that can win the game.

One of the first results about cop numbers given by Aigner and Fromme states that planar graphs have cop number at most 3 [1]. For graphs embeddable in a surface of genus g it is known that the cop number is at most linear in g and recent progress was made on improving the linear factor [8, 11, 24]. It is an outstanding open problem to determine which graphs embeddable in a surface of genus g have largest cop number. There are graphs of genus g with cop number at least $g^{\frac{1}{2}-o(1)}$, one such example are binomial random graphs $G_{n,p}$ with $p = \frac{2 \log n}{n}$ [5, 20]. The gap between the upper and lower bound is large and it is conjectured that the lower bound gives the right order of the cop number.

Conjecture 1 ([20, 22]). *Let S be a graph of genus g . Then $c(S) = O(\sqrt{g})$.*

Similarly, the cop win number for a surface of genus g is at most linear in g [21]. It was shown that for graphs of cop number at least 3 there exists a surface S of genus g with $c(S) \geq c(G)$ [21]. Therefore there are compact surfaces of genus g with cop win number at least $g^{\frac{1}{2}-o(1)}$. The following conjecture is a tough conjecture since it implies Conjecture 1.

Conjecture 2 ([22]). *Let S be a geodesic surface of genus g . Then $c(S) = O(\sqrt{g})$.*

¹The rules of the Cops and Robber game on graphs we define here are slightly different to standard rules, but they do not affect the outcome of the game on connected graphs.

Surprisingly, the upper bound from Conjecture 2 can be significantly improved for surfaces of genus g which are hyperbolic, that means they have constant curvature -1 . Since our results extend to higher-dimensional hyperbolic manifolds, we state the more general version.

Theorem 3. *If M is a compact hyperbolic manifold, then $c(M) = 2$.*

An important tool for determining upper bounds for cop numbers is the Isometric Path Lemma, which we use to establish Theorem 3 and to show the cop catch number for special surfaces of genus $g \geq 2$ is at most 6.

Lemma 4 ([1, 22]). *Let I be an isometric path starting at A and ending at B . Then one cop c can guard I after spending time equal to the length of I on the path to adjust himself.*

The Cops and Robber game on geodesic spaces is tightly related to continuous and discrete pursuit-evasion games on metric spaces. In continuous pursuit-evasion games the players make decisions at every point in the time interval $[0, \infty)$. For example, Besicovitch showed that in the Lion and Man game as introduced by Rado (see Littlewood's Miscellany [18]), the man can escape the lion when the game is played on a disk. Croft studied a variation of this game with multiple pursuers on higher dimensional balls [10] and Satimov and Kushkarov studied the game on the sphere [23].

The Discrete Lion and Man game is the Cops and Robber game where the agility function is constant, i.e. $\tau \equiv K$ for some constant K . It was shown that in the Discrete Lion and Man game the lion can catch the man on a disk, more generally, the lion can catch the man on any compact CAT(0)-space [4, 26]. While the Cops and Robber game is a discrete pursuit-evasion game, it captures the properties of the continuous game, in the sense that in the Cops and Robber game one cop cannot catch the robber when the game is played on a disk [15].

2 Proof Sketch of Theorem 3

We denote by \mathbb{H}^n the n -dimensional hyperbolic space. By the Killing-Hopf Theorem [13, 17] any hyperbolic manifold arises from a tessellation of hyperbolic space, for an example see Figure 2. More precisely, any hyperbolic manifold is isometric to \mathbb{H}^n/Γ where Γ is a group of isometries of \mathbb{H}^n acting freely and properly discontinuously. In order to show Theorem 3, we play the game in the covering space \mathbb{H}^n of the manifold. It was shown that if C is the covering space of a geodesic space X that locally preserves distances, then $c(X) \leq c(C)$ [15]. We will use the idea of the theorem in this proof. In order to simplify our exposition, we provide a sketch of the proof for Theorem 3 only for 2-dimensional manifolds which are surfaces. Let $s = \text{sys}(S)$ be the systolic girth of the hyperbolic surface S , which is the length of the smallest non-contractible curve.

To show that $c(S) > 1$ we play the game with one cop c and the robber r . The robber chooses the agility function $\tau \equiv \frac{s}{8}$ and initial positions such that $d(c^0, r^0) > \frac{s}{8}$. If $d(c^k, r^k) \geq \frac{3s}{8}$, then the robber does not move and $r^{k+1} = r^k$. If $d(c^k, r^k) < \frac{3s}{8}$, the robber

moves in the direction opposite to the cop's position, i.e. the shortest paths from r^k to c^k and r^k to r^{k+1} meet at r^k at angle π . To argue that such a position exists with the additional assumption that $d(r^{k+1}, c^k) = d(r^k, c^k) + \frac{s}{8}$, the Gauss-Bonnet Theorem can be applied, for details we refer to the full version of the paper [14]. In both cases the robber can stay at distance at least $\frac{s}{8}$ to the cop, which proves the lower bound.

We sketch the proof strategy for the upper bound. Let $D = \text{diam}(S)$, which is the largest distance between two points in S . The rounds are grouped into blocks, that is, we consider integers representing time steps $1 = t_0 < t_1 < t_2 < \dots$ such that $\sum_{k=t_i}^{t_{i+1}-1} \tau(k) \geq 30D$. For each time step k we choose a representation C_i^k, R^k of c_i^k, r^k in the covering space \mathbb{H}^2 . By definition of distance on a hyperbolic surface, $d_{\mathbb{H}^2}(C_i^k, R^k) \geq d_{\mathbb{H}^2}(c_i^k, r^k)$ ($i = 1, 2$). We show that the cops have a strategy such that $\inf_k d_{\mathbb{H}^2}(C_i^k, R^k) = 0$. At each time step t_i , the cop c_2 chooses a new representation $C_2^{t_i}$ of its position $c_2^{t_i}$ in the covering space, which means $d(C_2^{t_i-1}, C_2^{t_i})$ is possibly greater than $\tau(t_i)$ but we maintain that $d(c_2^{t_i-1}, c_2^{t_i}) \leq \tau(t_i)$. In between the time steps t_i, t_{i+1} we choose the representation of the cops and the robber such that they are coherent with the agility function, which means $d_{\mathbb{H}^2}(R^{k-1}, R^k) \leq \tau(k)$ and $d(C_i^{k-1}, C_i^k) \leq \tau(k)$ for $t_i < k < t_{i+1}$. Let $R^{t_i}, C_1^{t_i}$ be a copy of the robber's and cop's position in the covering space, such that their distance in the covering space is the same as in the surface. We consider the geodesic g_0 through $R^{t_i}, C_1^{t_i}$. The choice of the position $C_2^{t_i}$ for cop c_2 is such that it is close to the geodesic g_0 but sufficiently far from R^{t_i} , which we make more precise in the following.

Let P be the point on g_0 at distance $10D$ to R^{t_i} that is further away from $C_1^{t_i}$. Note that there is a copy $C_2^{t_i}$ of $c_2^{t_i}$ in the covering space which is at distance at most $D = \text{diam}(S)$ from P . We consider $h = o_{g_0}(C_2^{t_i})$, the orthogonal geodesic to g_0 through $C_2^{t_i}$.

Claim 1. $h \cap g_0$ is at distance between $9D$ and $11D$ from R^{t_i} and at distance at most D from $C_2^{t_i}$.

The strategy of cop c_2 is to chase the orthogonal projection of R^k on h . Note that by Claim 1 the distance from R^{t_i} to h is at least $9D$. Let B, B' be points on h at distance $8D$ from $B_0 := g_0 \cap h$, see Figure 1(a). The cop c_2 can guard the path from B to B' on h , since his distance to B_0 is at most D , so his distance to B, B' is at most $9D$, which is at least the distance from R^{t_i} to B, B' . Let $g_k = o_h(C_1^k)$ be the orthogonal geodesic to h through C_1^k , see Figure 1(b).

Suppose R^k, R^{k+1} are contained in the triangle defined by $B, h \cap g_k$ and C_1^k . Then we move cop c_1 towards B such that:

$$\text{The robber's position } R^{k+1} \text{ and } B \text{ are on the same side of } g_{k+1}. \tag{2}$$

The $(k + 1)$ -st position C_1^{k+1} of cop c_1 is

- (a) the point between C_1^k and B s.t. $d(C_1^{k+1}, C_1^k) = \tau(k + 1)$ if this step does not violate (2),
- (b) otherwise, the closest point to R^{k+1} on the geodesic $o_h(R^{k+1})$ (the orthogonal geodesic to h through R^{k+1}) which satisfies $d(C_1^{k+1}, C_1^k) = \tau(k + 1)$.

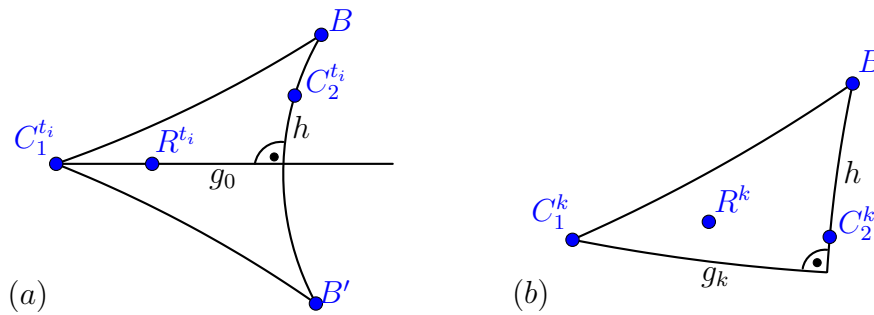


Figure 1: Figure (a) is a schematic drawing of the robber’s and the cops’ position at time step t_i ($1 \leq i$). Figure (b) is a schematic drawing of the robber’s and the cops’ position at time step $t_i < k < t_{i+1}$.

The strategy is similar if the robber moves towards B' . If the robber crosses $\overline{C_1^k B}$ or $\overline{C_1^k B'}$ for some $t_i < k < t_{i+1}$, then the cops’ strategy for the steps $t_i + 1, \dots, t_{i+1} - 1$ is simply to walk towards the robber. At step t_{i+1} the strategies reset.

Claim 2. *Either the robber gets caught by the cops or he crosses $\overline{C_1^t B}$ or $\overline{C_1^t B'}$ for some t with $t_i < t < t_{i+1}$. Further, for each $\varepsilon > 0$ there exists some $\delta = \delta(\varepsilon)$, such that if $d(R^{t_i}, C_1^{t_i}) \geq \varepsilon$, then*

$$d(R^t, C_1^t) - d(R^{t_i}, C_1^{t_i}) \geq \delta\varepsilon.$$

Claim 2 shows that the cop c_1 comes eventually ε -close to the robber, which means $d(R^k, C_1^k) < \varepsilon$ for some k . This is enough to show that the cops can win the game on S . For the proofs of Claims 1 and 2 we refer the reader to the full version of the paper [14]. \square

3 Catching the Robber

We define by $P(k, \theta)$ the regular k -gon in the Poincaré disk \mathcal{D} centred at $O = (0, 0)$ with angle θ at the vertices. We denote its vertices by v_1, \dots, v_k in counter-clockwise direction and let a_i be the (oriented) edge from v_i to v_{i+1} and a_i^{-1} be the reversed edge from v_{i+1} to v_i (we consider the indices modulo k). We are going to consider three standard hyperbolic surfaces for $g \geq 2$, where one of them is non-orientable.

- Let $S(g)$ be the orientable surface obtained from $P\left(4g, \frac{2\pi}{4g}\right)$ by identifying the (oriented) edges a_{4i-3} with a_{4i-1}^{-1} and a_{4i-2} with a_{4i}^{-1} for $i = 1, \dots, g$. The surface $S(2)$ is depicted in Figure 2.
- Let $S'(g)$ be the orientable surface obtained from $P\left(4g + 2, \frac{2\pi}{2g+1}\right)$ by identifying opposite (oriented) edges a_i, a_{i+2g+1}^{-1} for $i = 1, \dots, 2g$.

- Let $N(g)$ be the non-orientable surface obtained from $P\left(2g, \frac{2\pi}{2g}\right)$ by identifying the (oriented) edge a_{2i} with a_{2i+1} for $i = 1, \dots, g$.

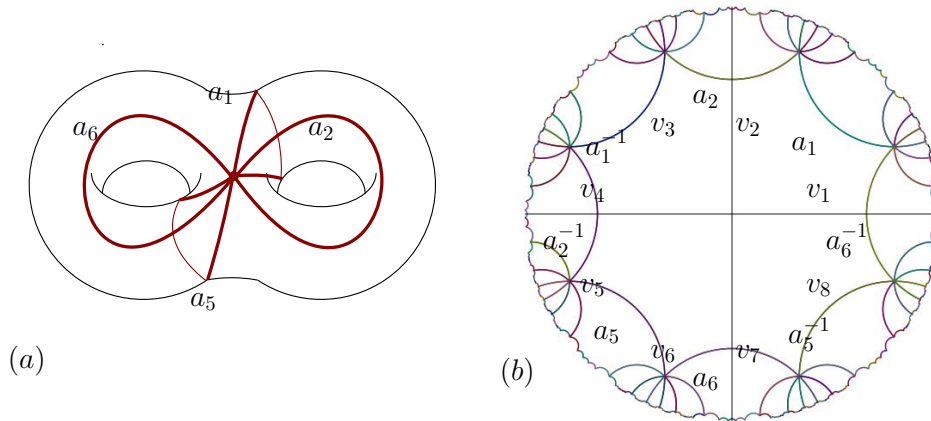


Figure 2: The surface $S(g)$ as a standard geometric model for the double torus with its fundamental domain $P\left(4g, \frac{2\pi}{4g}\right)$ in the Poincaré disk \mathcal{D} .

Lemma 5. *Suppose the robber is contained in a bounded convex polygon in the Poincaré disk \mathcal{D} where n cops guard the boundary of the polygon. Then these n cops can catch the robber.*

We can deduce the following theorem from Lemma 5.

Theorem 6. *If $g \geq 2$, then (a) $c_0(S(g)) \leq 5$, (b) $c_0(S'(g)) \leq 6$ and (c) $c_0(N(g)) \leq 4$.*

Proof. We will give the proof only for (a), the proof for the other surfaces is similar. Let O be the midpoint of the fundamental polygon $P\left(4g, \frac{2\pi}{4g}\right)$. We will play the game in the covering space and choose the player's positions such that they are in $P\left(4g, \frac{2\pi}{4g}\right)$. We will first use the cops c_1, c_2, c_3 to guard isometric paths. We start by moving cop c_1 to the isometric path $\overline{Ov_1}$, cop c_2 to the isometric path $\overline{Ov_5}$ and cop c_3 to the isometric path $\overline{Ov_9}$. By Lemma 4 we can assume that after a finite amount of time the cops guard the respective isometric paths. Now if the robber is in one of the triangles Ov_jv_{j+1} for some $1 \leq j \leq 4$, then the robber's moves are restricted to the specified triangles since $a_1 = a_3^{-1}$ and $a_2 = a_4^{-1}$. Similarly if the robber is contained in one of the triangles Ov_jv_{j+1} for some $5 \leq j \leq 8$ his moves are restricted to these triangles. If the robber is outside Ov_jv_{j+1} for some $1 \leq j \leq 8$, we move cop c_2 to the isometric path $\overline{Ov_{13}}$ and wait until he is guarding it. If the robber is in one of the triangles Ov_jv_{j+1} for $9 \leq j \leq 12$, his moves are restricted to these triangles. If the robber is not contained in one of these triangles we keep going for $i = 3, 4, \dots$ in the same way, moving the cop currently guarding $\overline{Ov_{1+4i}}$ to guard $\overline{Ov_{1+4(i+2)}}$ unless $i+2 = g$, in which case the robber is contained in one of $Ov_{4g-3}v_{4g-2}$, $Ov_{4g-2}v_{4g-1}$, $Ov_{4g-1}v_{4g}$ or $Ov_{4g}v_1$, cop c_1 guards Ov_1 and one of c_2 or c_3 guards Ov_{4g-3} .

We assume without loss of generality that cop c_1 guards $\overline{Ov_1}$, cop c_2 guards $\overline{Ov_5}$ and the robber is in one of the triangles Ov_jv_{j+1} for some $1 \leq j \leq 8$. Cop c_3, c_4, c_5 will guard $\overline{Ov_2}, \overline{Ov_3}, \overline{Ov_4}$, respectively. Now the robber is captured in either $R_1 = Ov_1v_2 \cup Ov_3v_4$ or $R_2 = Ov_2v_3 \cup Ov_4v_5$. The regions R_1 and R_2 can be embedded in the covering space \mathcal{D} such that they form a quadrilateral which is guarded by four of the cops. By Lemma 5 we can now catch the robber. \square

References

- [1] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Appl. Math.*, 8(1):1–11, 1984.
- [2] James W Anderson. Convexity, area, and trigonometry. In *Hyperbolic Geometry*, pages 111–151. Springer, 1999.
- [3] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Springer Science & Business Media, 1992.
- [4] Andrew Beveridge and Yiqing Cai. Two-dimensional pursuit-evasion in a compact domain with piecewise analytic boundary. *ArXiv e-prints*, May 2015.
- [5] Béla Bollobás, Gábor Kun, and Imre Leader. Cops and robbers in a random graph. *Journal of Combinatorial Theory, Series B*, 103(2):226–236, 2013.
- [6] Anthony Bonato. *An invitation to pursuit-evasion games and graph theory*, volume 97 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2022.
- [7] Anthony Bonato and Richard J. Nowakowski. *The game of cops and robbers on graphs*, volume 61 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2011.
- [8] Nathan Bowler, Joshua Erde, Florian Lehner, and Max Pitz. Bounding the cop number of a graph by its genus. *SIAM J. Discrete Math.*, 35(4):2459–2489, 2021.
- [9] Peter Bradshaw. A proof of the meyniel conjecture for abelian cayley graphs. *Discrete Mathematics*, 343(1):111546, 2020.
- [10] H. T. Croft. 'Lion and man': A postscript. *J. Lond. Math. Soc.*, 39:385–390, 1964.
- [11] Joshua Erde and Florian Lehner. Improved bounds on the cop number of a graph drawn on a surface. In *Extended Abstracts EuroComb 2021*, pages 111–116. Springer, 2021.
- [12] Peter Frankl. Cops and robbers in graphs with large girth and Cayley graphs. *Discrete Appl. Math.*, 17(3):301–305, 1987.

- [13] Heinz Hopf. Zum Clifford-Kleinschen Raumproblem. *Math. Ann.*, 95(1):313–339, 1926.
- [14] Vesna Iršič, Bojan Mohar, and Alexandra Wesolek. Cops and robber on hyperbolic manifolds. *In preparation*.
- [15] Vesna Iršič, Bojan Mohar, and Alexandra Wesolek. Cops and robber game in higher-dimensional manifolds with spherical and Euclidean metric. *C. R. Math. Acad. Sci. Soc. R. Can.*, 44(3):50–68, 2022.
- [16] Gwenaël Joret, Marcin Kamiński, and Dirk Oliver Theis. The cops and robber game on graphs with forbidden (induced) subgraphs. *Contrib. Discrete Math.*, 5(2):40–51, 2010.
- [17] Wilhelm Killing. Ueber die Clifford-Klein'schen Raumformen. *Math. Ann.*, 39(2):257–278, 1891.
- [18] J. E. Littlewood. *Littlewood's miscellany*. Cambridge University Press, Cambridge, 1986. Edited and with a foreword by Béla Bollobás.
- [19] Tomasz Łuczak and Paweł Prałat. Chasing robbers on random graphs: zigzag theorem. *Random Structures & Algorithms*, 37(4):516–524, 2010.
- [20] Bojan Mohar. Notes on cops and robber game on graphs. *ArXiv e-prints*, October 2017.
- [21] Bojan Mohar. Min-max theorem for the game of cops and robber on geodesic spaces. *ArXiv e-prints*, December 2021.
- [22] Bojan Mohar. The game of cops and robber on geodesic spaces. *ArXiv e-prints*, May 2022.
- [23] N. Yu. Satimov and A. Sh. Kuchkarov. On the solution of a model differential pursuit-evasion game on a sphere. *Uzb. Mat. Zh.*, 2000(1):45–50, 2000.
- [24] Bernd SW Schröder. The copnumber of a graph is bounded by $\lceil 3/2 \text{ genus}(g) \rceil + 3$. In *Categorical perspectives*, pages 243–263. Springer, 2001.
- [25] Stothers, Wilson. Hyperbolic geometry. <http://www.maths.gla.ac.uk/wws/cabripages/hyperbolic/hyperbolic0.html>. [Online; accessed 16-January-2023].
- [26] Olga Yufereva. Lion and man game in compact spaces. *Dyn. Games Appl.*, 9(1):281–292, 2019.