

# STACK AND QUEUE NUMBERS OF GRAPHS REVISITED

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## Abstract

A long-standing question of the mutual relation between the stack and queue numbers of a graph, explicitly emphasized by Dujmović and Wood in 2005, was “half-answered” by Dujmović, Eppstein, Hickingbotham, Morin and Wood in 2022; they proved the existence of a graph family with the queue number at most 4 but unbounded stack number. We give an alternative very short, and still elementary, proof of the same fact.

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## 1 Introduction

The graph parameters called stack and queue numbers relate to *linear layouts* (i.e, linear vertex orderings, usually of additional “nice” properties) of graphs, and have found numerous applications in theoretical computer science since then. The parameters were formally introduced by Heath, Leighton, and Rosenberg in [6,7], and their implicit question of whether the stack number of a graph is bounded in terms of its queue number, or vice versa, was subsequently emphasized by Dujmović and Wood in [3]. Quite recently, in 2022, Dujmović, Eppstein, Hickingbotham, Morin and Wood gave in [2] a negative answer to one half of the question; they proved the existence of a graph family with the queue number at most 4 but unbounded stack number (while it remains an open problem whether there exists a family of bounded stack number and unbounded queue number).

We give the basic definitions. Consider a graph  $G$  and a strict linear order  $\prec$  on its vertex set  $V(G)$ . Two edges  $xx', yy' \in E(G)$  with  $x \prec x'$  and  $y \prec y'$  are said to  $\prec$ -cross if

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$x \prec y \prec x' \prec y'$  or  $y \prec x \prec y' \prec x'$ , and to  $\prec$ -nest if  $x \prec y \prec y' \prec x'$  or  $y \prec x \prec x' \prec y'$ . See Figure 1. The *stack number*  $\text{sn}(G)$  (*queue number*  $\text{qn}(G)$ ) of a graph  $G$  is the minimum integer  $k$  such that there exist a linear order  $\prec$  of  $V(G)$  and a colouring of the edges of  $G$  by  $k$  colours such that no two edges of the same colour  $\prec$ -cross ( $\prec$ -nest, resp.). The corresponding order  $\prec$  together with the colouring is called a  $k$ -stack ( $k$ -queue) layout of  $G$ .



Figure 1: Edges  $xx'$  and  $yy'$  that (a)  $\prec$ -cross, and (b)  $\prec$ -nest.

In fact, a notion equal (modulo a negligible technical detail) to the stack number was known long before as the *book thickness* (or page number), see Persinger [8] and Atneosen [1].

To state the main result of [2], we define the following special graph  $H_n$ : the vertex set is  $V(H_n) = \{1, \dots, n\}^2$ , and  $uv \in E(H_n)$  where  $u = [a, b] \in V(H_n)$  and  $v = [c, d] \in V(H_n)$ , if and only if  $|a - c| + |b - d| = 1$  or  $a - c = b - d \in \{-1, 1\}$ . Note that  $H_n$  is the plane dual of the hexagonal (“honeycomb”) grid, and see an illustration in Figure 2.

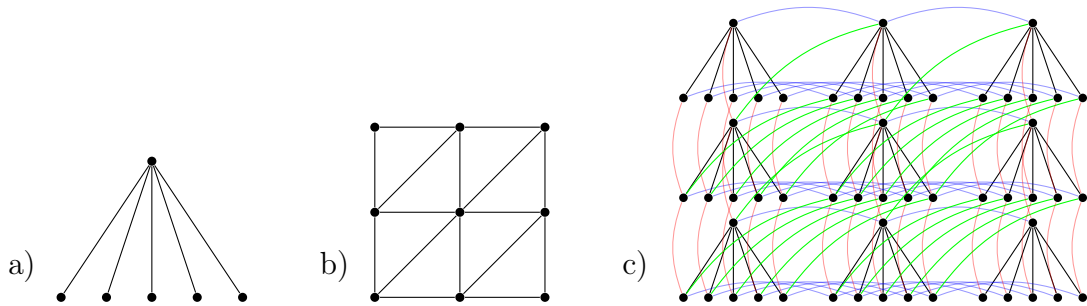


Figure 2: (a) The star  $S_5$ , (b) the graph  $H_3$ , and (c) their Cartesian product  $S_5 \square H_3$ . The four edge colours illustrate a queue layout for  $S_5 \square H_3$ .

Recall that  $S_n$  is the star with  $n$  leaves, and that  $G_1 \square G_2$  denotes the Cartesian product of two graphs  $G_1$  and  $G_2$ . Dujmović et al. [2] showed that, for all integers  $a, n > 0$  and the Cartesian product  $G = S_a \square H_n$ , we have  $\text{qn}(G) \leq 4$ . In fact, they noted that every  $H_n$  admits a so-called *strict 3-queue layout*, which “adds up” with a trivial 1-queue layout of  $S_a$  over Cartesian product by Wood [12]. Their main result reads:

**Theorem 1** (Dujmović et al. [2]). *For every integer  $s$ , and for  $a, n > 0$  which are sufficiently large with respect to  $s$ , the Cartesian product  $G := S_a \square H_n$  is of stack number at least  $s$ .*

Our contribution is to give a very short simplified proof of Theorem 1 (based in parts on the ideas from [2], but also eliminating some rather long fragments of the former proof).

## 2 Proof of Theorem 1

We will use some classical results, the first two of which are truly folklore.

**Proposition 2** (Ramsey [9]). *For all integers  $r, s > 0$  there exists  $R = R(r, s)$  such that for any assignment of two colours red and blue to the edges of the complete graph  $K_R$ , there is a red clique on  $r$  vertices or a blue clique on  $s$  vertices in it.*

**Proposition 3** (Erdős–Szekeres [4]). *For given integers  $r, s > 0$ , any sequence of distinct elements of a linearly ordered set of length more than  $r \cdot s$  contains an increasing subsequence of length  $s + 1$  or a decreasing subsequence of length  $r + 1$ .*

**Proposition 4** (Gale [5]). *Consider a dual hexagonal grid  $H_n$  as above. For any assignment of two colours to the vertices of  $H_n$ , there exists a monochromatic path on  $n$  vertices.*

Consider for the rest any fixed stack layout of the graph  $G$  of Theorem 1, with the linear order  $\prec$  on the vertex set  $V(G)$ . Recall that  $V(G) = \{(u, p) : u \in V(S_a), p \in V(H_n)\}$ .

**Lemma 5.** *Let  $L$  be the set of leaves of  $S_a$ , and let  $b = a^{-m}$  where  $m = 2^{n^2-1}$ . There is a subsequence  $(u_1, \dots, u_b)$  in the set  $L$  of length  $b$  such that for each  $p \in V(H_n)$ , either  $(u_1, p) \prec (u_2, p) \prec \dots \prec (u_b, p)$ , or  $(u_1, p) \succ (u_2, p) \succ \dots \succ (u_b, p)$ .*

*Proof.* Let  $V(H_n) = \{p_1, \dots, p_{n^2}\}$  be the vertices of  $H_n$ . Start with the permutation  $\sigma_1 = (u_{i[1,1]}, \dots, u_{i[1,a_1=a]})$  of  $L$  such that  $(u_{i[1,1]}, p_1) \prec \dots \prec (u_{i[1,a_1]}, p_1)$ . By Proposition 3, for each  $j \in \{2, \dots, n^2\}$ , the sequence  $\sigma_{j-1}$  contains a subsequence  $\sigma_j = (u_{i[j,1]}, \dots, u_{i[j,a_j]})$  such that  $a_j \geq \sqrt{a_{j-1}}$ , and  $(u_{i[j,1]}, p_j) \prec \dots \prec (u_{i[j,a_j]}, p_j)$  or  $(u_{i[j,1]}, p_j) \succ \dots \succ (u_{i[j,a_j]}, p_j)$ . By simple calculus, we get  $a_{n^2} \geq a_1^m = b$  which is the desired outcome.  $\square$

Let  $S_b \subseteq S_a$  be the (specific) substar of  $S_a$  defined by the subset of leaves  $\{u_1, \dots, u_b\}$  (of Lemma 5). Colour every vertex  $p \in V(H_n)$  red if  $(u_1, p) \prec \dots \prec (u_b, p)$ , and colour  $p$  blue otherwise. From this and Proposition 4, we immediately obtain:

**Corollary 6.** *There is a subgraph  $Q \subseteq H_n$ , being a path on  $n$  vertices, such that, without loss of generality,  $(u_1, q) \prec \dots \prec (u_b, q)$  holds for every vertex  $q \in V(Q)$ .*  $\square$

Define  $X \subseteq G$  to be the subgraph induced on the vertex set  $V(S_b) \times V(Q)$ , i.e.,  $X = S_b \square Q$ , and denote by  $\mathcal{R}$  the set of paths  $R_i \subseteq X$  induced on  $\{u_i\} \times V(Q)$  for  $i = 1, \dots, b$ . We extend  $\prec$  to a partial order on  $\mathcal{R}$  as follows; for  $R_i, R_j \in \mathcal{R}$ , we have  $R_i \prec R_j$ , if and only if  $u \prec w$  for all  $u \in V(R_i)$  and  $w \in V(R_j)$ . We say that  $R_i$  and  $R_j$  are  $\prec$ -separated if  $R_i \prec R_j$  or  $R_i \succ R_j$ , and that  $R_i$  and  $R_j$  are  $\prec$ -crossing if there exist edges  $e \in E(R_i)$  and  $f \in E(R_j)$  such that  $e, f$   $\prec$ -cross. The following is simple but crucial:

**Lemma 7.** *Every two distinct paths  $R_i, R_j \in \mathcal{R}$  are either  $\prec$ -crossing, or  $\prec$ -separated.*

*Proof.* Assume the contrary; up to symmetry meaning that all edges of  $R_i$  are nested in some edge  $e_2 = \{(u_j, q), (u_j, q')\} \in E(R_j)$ . Then, in particular,  $e_1 = \{(u_i, q), (u_i, q')\} \in E(R_i)$  is nested in  $e_2$ , and so  $(u_j, q) \prec (u_i, q)$  and  $(u_j, q') \succ (u_i, q')$ . This contradicts Corollary 6.  $\square$

**Corollary 8.** *For all integers  $c, d$  and  $n$ , and for  $b = |\mathcal{R}|$  sufficiently large with respect to  $c, d$ , we have that  $\mathcal{R}$  contains at least  $c$  pairwise  $\prec$ -separated or  $d$  pairwise  $\prec$ -crossing paths.*

*Proof.* Imagine a pair of paths  $\{R_i, R_j\} \subseteq \mathcal{R}$  coloured red if  $R_i, R_j$  are  $\prec$ -crossing, and blue if they are  $\prec$ -separated. With respect to Lemma 7, we apply Proposition 2 with  $b \geq R(c, d)$ .  $\square$

We finish as follows.

*Proof of Theorem 1.* Respecting the above definition of the set of paths  $\mathcal{R}$  in  $G$ , we branch into the two cases determined by Corollary 8.

**Case I.** There are  $c$  pairwise  $\prec$ -separated paths in  $\mathcal{R}$ .

Without loss of generality, let these paths be  $R_1 \prec \dots \prec R_c$ . For the root  $t$  of  $S_b$ , label the  $n$  vertices of the set  $\{t\} \times V(Q) \subseteq V(X)$  by  $t_1 \prec \dots \prec t_n$ . There are two subcases.

- $R_{\lfloor c/2 \rfloor} \prec t_{\lceil n/2 \rceil}$ . For each  $i = 1, \dots, \min(\lfloor c/2 \rfloor, \lceil n/2 \rceil)$ , pick an edge of  $X$  from  $t_{\lceil n/2 \rceil + i - 1}$  to  $V(R_i)$  (which exist since  $R_i$  hits every copy of  $S_b$  in  $X$  by the definition). We have got  $\min(\lfloor c/2 \rfloor, \lceil n/2 \rceil)$  edges in  $X$  that pairwise  $\prec$ -cross, as in Figure 3.

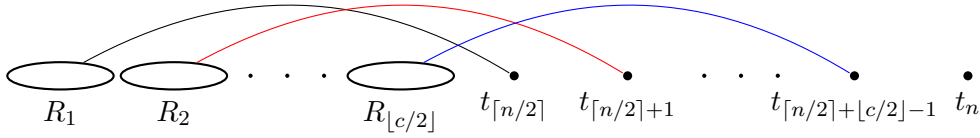


Figure 3: Case I, where  $R_{\lfloor c/2 \rfloor} \prec t_{\lceil n/2 \rceil}$  and  $\lceil n/2 \rceil > \lfloor c/2 \rfloor$ .

- $t_{\lceil n/2 \rceil} \prec R_{\lfloor c/2 \rfloor + 1}$  (note that  $t_{\lceil n/2 \rceil}$  may be “ $\prec$ -nested” in  $R_{\lfloor c/2 \rfloor}$ ). This is symmetric to the previous, and we get  $\min(\lceil c/2 \rceil, \lceil n/2 \rceil)$  pairwise  $\prec$ -crossing edges in  $X$  between vertices of  $R_{\lfloor c/2 \rfloor + 1}, \dots, R_c$  and  $s_1, \dots, s_{\lceil n/2 \rceil}$ .

**Case II.** There are  $d$  pairwise  $\prec$ -crossing paths in  $\mathcal{R}$ .

Pick any path  $R_0$  out of these  $d$  paths. In  $Z := \bigcup_{R \in \mathcal{R}, R \neq R_0} E(R)$  there are at least  $d - 1$  edges which  $\prec$ -cross some edge of  $R_0$ , and so at least  $(d - 1)/n$  of them cross the same edge  $e \in E(R_0)$ . Having  $e = u_1 u_2$ ,  $u_1 \prec u_2$ , and  $f = v_1 v_2 \in E(X)$  such that  $e$  and  $f$   $\prec$ -cross, we say that  $v_1$  is the *inside* vertex of  $f$  if  $u_1 \prec v_1 \prec u_2$ , and then  $v_2$  is the *outside* vertex. By the pigeonhole principle, there is a set  $Z' \subseteq Z$  of  $d' = |Z'| \geq (d - 1)/n^2$  edges  $\prec$ -crossing  $e$  such that their inside vertices belong to the same copy of  $S_b$  in  $X$ .

The outside vertices of the edges of  $Z'$  belong to at most two other copies of  $S_b$  in  $X$  (determined by a neighbourhood in the path  $Q$ ), and each is before or after  $e$  in  $\prec$ . By the pigeonhole principle again, and without loss of generality, there is a set  $Z'' \subseteq Z'$  of size  $|Z''| \geq \frac{1}{2} \cdot \frac{1}{2}d' = d'/4$ , such that also the outside vertices of the edges of  $Z''$  belong to the same copy of  $S_b$  in  $X$ , and they all lie after  $e$  in  $\prec$ . See Figure 4. Moreover, by Corollary 6 (the ordering claimed therein), the edges in  $Z''$  must pairwise  $\prec$ -cross.

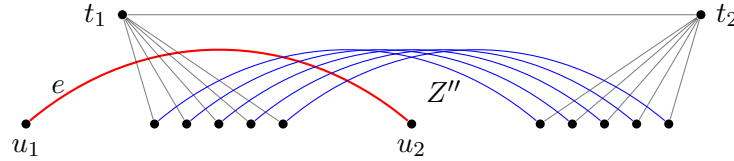


Figure 4: Case II, with emphasized edge  $e$ , blue pairwise-crossing edges of  $Z''$ , and  $t_1, t_2$  being two copies of the root of  $S_b$ .

To finish the proof, we set  $n = 2s$  and  $a = R(2s, 4n^2s + 1)^m$  where  $m = 2^{n^2-1}$ . Then in Lemma 5 we get  $b = R(2s, 4n^2s + 1)$ , and in Corollary 8 we have  $c = 2s$  and  $d = 4n^2s + 1$ . In Case I, we then obtain at least  $\min(\lfloor c/2 \rfloor, \lceil n/2 \rceil) = s$  edges of  $X \subseteq G$  that pairwise  $\prec$ -cross. In Case II, it is at least  $d'/4 = (d - 1)/(4n^2) = s$  such pairwise  $\prec$ -crossing edges, too. Edges that pairwise  $\prec$ -cross obviously must receive distinct colours. A valid stack layout based on  $\prec$  hence needs at least  $s$  colours, and since  $\prec$  has been arbitrary for the graph  $G$ , we finally conclude that  $\text{sn}(G) \geq s$ .  $\square$

### 3 Conclusion

We have provided a short elementary proof of Theorem 1. Although the original proof in [2] is not very long or difficult, by carefully rearranging the arguments we have succeeded in eliminating some technical steps of the proof in [2] and, in particular, resolved the case of pairwise crossing paths in a direct short way. Briefly explaining, our proof skips initial technical parts of [2] preceding the use of Proposition 3 (Erdős-Szekeres) and readily applies Proposition 3 and Proposition 4 in a way similar to [2], and then it concludes by Proposition 2 (Ramsey) in which both outcomes straightforwardly lead to a large set of pairwise crossing edges, thus avoiding other technical steps needed in [2] mainly at the end of the arguments.

The presented proof is based on the Bachelor's thesis of the second author [10, 11].

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