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Twin-width of Planar Graphs; a Short Proof

(Extended abstract)

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Abstract

The fascinating question of the maximum value of twin-width on planar graphs is nowadays not far from a final resolution; there is a lower bound of 7 coming from a construction by Král' and Lamaison [arXiv, September 2022], and an upper bound of 8 by Hliněný and Jedelský [arXiv, October 2022]. The upper bound (currently best) of 8, however, is rather complicated and involved. We give a short and simple self-contained proof that the twin-width of planar graphs is at most 11.

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1 Introduction

The structural parameter twin-width was introduced in 2020 by Bonnet, Kim, Thomassé and Watrigant [2]. We consider it only for simple graphs (instead of general binary relational structures).

A *trigraph* is a simple graph G in which some edges are marked as *red*, and with respect to the red edges only, we naturally speak about *red neighbours* and *red degree* in G. However, when speaking about edges, neighbours and/or subgraphs without further specification, we count both ordinary and red edges together as one edge set. The edges of G which are not red are sometimes called (and depicted) black for distinction. For a pair of (possibly not adjacent) vertices $x_1, x_2 \in V(G)$, we define a *contraction* of the pair x_1, x_2 as the operation creating a trigraph G' which is the same as G except that x_1, x_2 are replaced with a new vertex x_0 (said to *stem from* x_1, x_2) such that:

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- the (full) neighbourhood of x_0 in G' (i.e., including the red neighbours), denoted by $N_{G'}(x_0)$, equals the union of the neighbourhoods $N_G(x_1)$ of x_1 and $N_G(x_2)$ of x_2 in G except x_1, x_2 themselves, that is, $N_{G'}(x_0) = (N_G(x_1) \cup N_G(x_2)) \setminus \{x_1, x_2\}$, and
- the red neighbours of x_0, denoted here by $N^r_{G'}(x_0)$, inherit all red neighbours of x_1 and of x_2 and add those in $N_G(x_1)\Delta N_G(x_2)$, that is, $N^r_{G'}(x_0) = \big(N^r_G(x_1) \cup N^r_G(x_2) \cup (N_G(x_1)\Delta N_G(x_2))\big) \setminus \{x_1, x_2\}$, where Δ denotes the symmetric set difference.

A *contraction sequence* of a trigraph G is a sequence of successive contractions turning G into a single vertex, and its *width* d is the maximum red degree of any vertex in any trigraph of the sequence. We also then say that it is a d-contraction sequence of G. The *twin-width* of a trigraph G is the minimum width over all possible contraction sequences of G. In other words, a graph has twin-width at most d, iff it admits a d-contraction sequence.

After the first implicit (and astronomical) upper bounds on the twin-width of planar graphs, e.g. [2], we have seen a stream of improving explicit bounds [1,3,4,6], culminating with the current best upper bound of 8 by Hliněný and Jedelský [5]. This is complemented with a nearly matching lower bound of 7 by Král' and Lamaison [7], but the right maximum value (7 or 8?) is still an open question.

It comes without surprise that the gradually improving upper bounds have required stronger and more involved arguments, and the best ones are not easy to read for non-experts. In this paper, we take the opposite route; we give a slightly worse bound with a self-contained proof which is as short and simple as possible with the current knowledge:

Theorem 1. *The twin-width of any simple planar graph is at most* 11*.*

Due to page limits, some full proofs are left for the preprint version arXiv:2302.08938.

2 Layered Skeletal Trigraphs

We start with the key concept of our proof – of a "splendid layered skeletal trigraph".

We use standard terminology of graph theory, and assume every graph to be simple (without loops and multiple edges). A *BFS tree* of a graph G is a spanning tree defined by a run of the breadth-first-search algorithm on G.

For a (tri)graph G, an ordered partition $\mathcal{L} = (L_0, L_1, \ldots)$ of $V(G)$ is called a *layering of* G if, for every edge $\{v, w\}$ of G with $v \in L_i$ and $w \in L_j$, we have $|i - j| \leq 1$. For example, every BFS tree $T \subseteq G$ with the root r naturally defines a layering; $L_0 = \{r\}$, and L_i for $i > 0$ consisting of all vertices of G at graph distance i from r.

If $T \subseteq G$ is a rooted tree (e.g., a BFS tree), a path $P \subseteq G$ is called T*-vertical* if $P \subseteq T$ is a subpath of some leaf-to-root path of T.

Definition 2 (Skeletal trigraph)**.** Let H be a trigraph and $S \subseteq H$ a 2-connected planar subgraph such that all edges of H induced by $V(S)$ are black (note; including the edges not in $E(S)$). Fix a plane embedding of S, and call S a *plane skeleton of* H. Further, consider a *face assignment* of H in S in which every connected component H_0 of $H - V(S)$

is assigned to some face ϕ of S, such that all neighbours of H_0 in $V(S)$ belong to ϕ. Denote by U_ϕ the union of the vertex sets of all components assigned to ϕ in this assignment.

If H and S satisfy the previous conditions for some face assignment, we call (H, S) a *skeletal trigraph*, and if $\mathcal{L}$ is a layering of H, then $(H, S, \mathcal{L})$ is a *layered skeletal trigraph.*

Definition 3 (Splendid layered skeletal trigraph)**.** Consider a layered skeletal trigraph $(H, S, \mathcal{L})$ as in Definition 2, and a face ϕ of S. We say that ϕ is *empty* if $U_\phi = \emptyset$ (i.e., if no connected component of $H - V(S)$ is assigned to ϕ), that ϕ is *reduced* if $|U_\phi \cap L_i| \leq 1$ holds for every layer $L_i \in \mathcal{L}$, and that ϕ is *rich* if $|U_\phi \cap L_i| \leq 3$ holds for every $L_i \in \mathcal{L}$.

A layered skeletal trigraph $(H, S, \mathcal{L})$ is *splendid* if either $S = \emptyset$ and $|V(H) \cap L_i| \leq 4$ holds for all $L_i \in \mathcal{L}$, or $S \neq \emptyset$ and the following conditions are satisfied:

a) At most one face of the plane skeleton S is rich, and all other faces of S are empty or reduced. Every empty face of S is a triangle.
b) There exists a BFS tree $T \subseteq S$ of the skeleton S such that:
   - The layering defined by T in S is equal to the restriction of $\mathcal{L}$ to $V(S)$.
   - For every non-empty face ϕ of S, bounded by a cycle $C \subseteq S$, there exists an edge $e \in E(C)$ such that $C - e$ is the union of two T-vertical paths intersecting in one vertex $u \in V(C)$. Note that such u must be unique, and we call u the *sink of* ϕ.
c) For every non-empty face ϕ of S, and u the sink of ϕ; if $u \in L_i \in \mathcal{L}$, then all vertices of $U_\phi \cup V(C - u)$ belong to $L_{i+1} \cup L_{i+2} \cup \ldots$, and there is a black edge in H (but *no* red edge) from u to each vertex of $U_\phi \cap L_{i+1}$.
d) Assume ϕ is a rich face of S bounded by C. For every i such that $L_i \in \mathcal{L}$, every vertex v in $X := (U_\phi \cup V(C)) \cap L_i$ has in H at most 3 red edges into other vertices of X and at most 4 red edges into $U_\phi \cap (L_{i-1} \cup L_{i+1})$ (note; no $V(C)$ in the latter expression). Moreover, if $|U_\phi \cap L_{i+1}| > 1$, then $v \in X$ has at most 2 red edges into $U_\phi \cap L_{i-1}$.

Definition 3 is illustrated, with comments, in Figure 1.

The core of the paper is in the following two claims which follow directly from Definition 3. While the first one is easy and its proof is skipped here, a proof of the second one is sketched in the next section.

Lemma 4.* *Every splendid layered skeletal trigraph has maximum red degree at most* 11*.*

Lemma 5. *Every splendid layered skeletal trigraph admits an* 11*-contraction sequence.*

We now show how the claim implies our main result.

Proof of Theorem 1. Given a planar graph G, we fix any plane embedding of G. We construct a plane triangulation G^+ from G by adding new vertices to every face of G and connecting them to vertices of this face. Then G^+ is 2-connected. Choosing an arbitrary BFS tree of G^+, we take the layering $\mathcal{L} = (L_0, L_1, \ldots)$ of G^+ naturally defined by T. Then, trivially, $(G^+, G^+, \mathcal{L})$ is a splendid layered skeletal trigraph, and hence G^+ admits an 11-contraction sequence by Lemma 5. Restricting this sequence only to the contractions of pairs from $V(G)$ we, again trivially, obtain an 11-contraction sequence of G. □

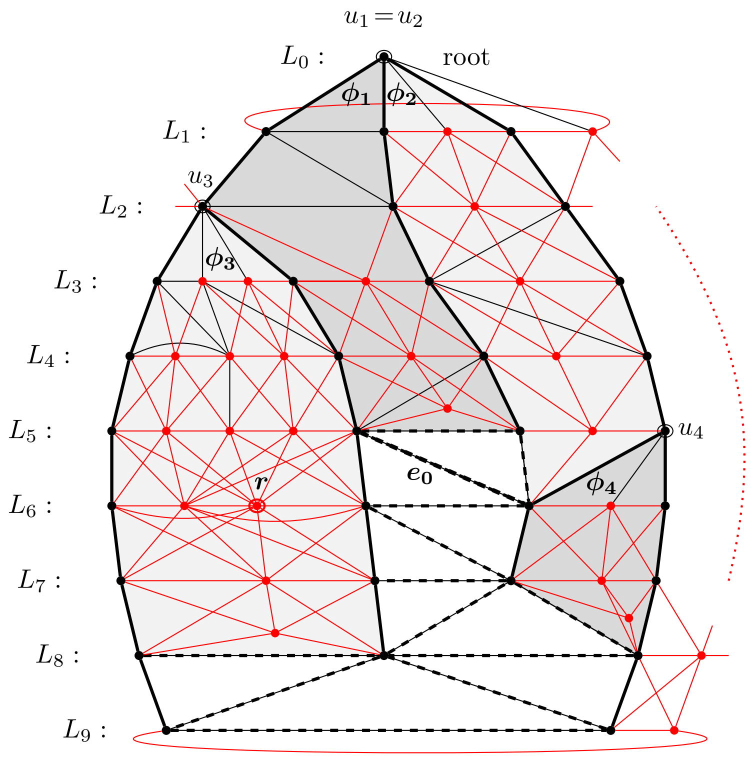

Figure 1: A picture of a splendid layered skeletal trigraph $(H, S, \mathcal{L})$, in which the skeleton S is depicted with black vertices and thick black edges such that the associated BFS tree $T \subseteq S$ is drawn with thick solid edges and the edges of $E(S) \setminus E(T)$ are thick dashed. T has its root at the top and its (ten) BFS layers are organized horizontally in the picture. There are four bounded non-empty faces in S, denoted by $\phi_1, \phi_2, \phi_3, \phi_4$ (with corresponding sinks u_1, u_2, u_3, u_4), and emphasized with gray shade. The unbounded face of S is also non-empty, but it is only sketched in the picture. There is one rich face in (H, S), namely ϕ_3, and it contains a red vertex r (emphasized with a circle around) that achieves the maximum red degree 11 allowed by Definition 3.

3 Proof of Lemma 5, a Sketch

Our proof starts with an auxiliary claim whose straightforward proof is skipped here.

Lemma 6.* *Let G be a 2-connected plane graph, and $T \subseteq G$ a BFS tree of G. Assume T that has at least 3 leaves, and that for every facial cycle C of G, we have $|E(C) \setminus E(T)| = 1$ or C is a triangle. Then there exists an edge $e \in E(G) \setminus E(T)$ such that, for the unique cycle $D_e \subseteq T + e$, one of the two faces of D_e contains (in its strict interior) precisely one leaf of T and not the root of T.*

For a proof of Lemma 5, consider a splendid layered skeletal trigraph $(H, S, \mathcal{L})$. For start, the maximum red degree of H is at most 11 by Lemma 4. For the rest of a sought 11-contraction sequence of H, we proceed by induction on $|V(H)| + |V(S)|$.

If the skeleton is empty $S = \emptyset$, then we pick the highest index i such that $V(H) \cap L_i \neq \emptyset$ and straightforwardly contract from layer i down by induction. If $S \neq \emptyset$, all faces of S are reduced (or empty), and the BFS tree $T \subseteq S$ from Definition 3.b has at most 2 leaves, we get that T consists of at most two T-vertical paths, and that S has at most two non-empty faces by Definition 3. b. Since the two faces are reduced, every layer of $\mathcal{L}$ contains at most $2 + 2 = 4$ vertices. So, $(H, S' = \emptyset, \mathcal{L})$ is also a splendid layered skeletal trigraph (note; no contraction happend) and we continue as before again by induction.

For all other cases, with a nonempty skeleton $S \neq \emptyset$, we branch as follows.

Case 1. The skeleton S has all faces empty. Then $S = H$ since S is a plane triangulation by Definition 3.a. Considering the BFS tree $T \subseteq S$ from Definition 3.b, we apply Lemma 6 and get e and cycle $D_e \subseteq T + e \subseteq H$. Let Q be the maximal T-vertical path starting in x and not hitting D_e. We set $S' := S - V(Q)$, using that the layered skeletal trigraph $(H, S', \mathcal{L})$ is splendid again, and we finish by induction.

Case 2. The skeleton S has a face ϕ which is neither empty nor reduced. Then ϕ is a rich face, and let j be the largest index such that $|U_\phi \cap L_j| > 1$ for $L_j \in \mathcal{L}$. We contract any two vertices of $U_\phi \cap L_j$ in H, creating a layered skeletal trigraph $(H', S, \mathcal{L}')$. For an illustration, see the face $\phi = \phi_3$ in Figure 1 in which the trigraph resulted by a contraction of two vertices from $U_{\phi_3} \cap L_6$ into the emphasized vertex r. We show that $(H', S, \mathcal{L}')$ conforms to Definition 3, and then apply induction.

Case 3. The skeleton S has all faces reduced (and some non-empty). As in Case 1, we apply Lemma 6 and get e and cycle $D_e \subseteq T + e \subseteq S$, and the path $Q \subseteq S$ in the interior of D_e. The interior of D_e contains at most two non-empty faces ϕ_1 and ϕ_2 of S. The considered case can be illustrated in Figure 1 (ignoring for now that the face ϕ_3 is not reduced) with the edge $e = e_0$. In general, there can be more than one empty faces of the skeleton S enclosed by D_{e_0}. We again set $S' := S - V(Q)$ and consider the layered skeletal trigraph $(H, S', \mathcal{L})$ with the (new) non-empty face ϕ bounded by D_e, which can be shown rich. Consequently, $(H, S', \mathcal{L})$ conforms to Definition 3, and we again finish by induction with it.

The whole proof, modulo straightforwad details regarding Definition 3, is now done. □

4 Conclusion

We have provided a short self-contained proof of Theorem 1. While the proved bound is not the best currently possible, the proof given here is way simpler than those in [4, 5]. While sacrificing a bit of simplicity of the given proof, we can also give a better upper bound of 9 (thus matching [4]), but we are so far not sure whether a similarly simplified proof can be given for the upper bound of 8 as in [5].

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