

PERMUTATION FLIP PROCESSES

(EXTENDED ABSTRACT)

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Abstract

We introduce a broad class of stochastic processes on permutations which we call flip processes. A single step in these processes is given by a local change on a randomly chosen fixed-sized tuple of the domain. We use the theory of permutons to describe the typical evolution of any such flip process $\pi_0, \pi_1, \pi_2, \dots$ started from any initial permutation $\pi_0 \in \text{Sym}(n)$. More specifically, we construct trajectories $\Phi : \mathfrak{P} \times [0, \infty) \rightarrow \mathfrak{P}$ in the space of permutons with the property that if π_0 is close to a permuton γ then for any $T > 0$ with high probability π_{Tn} is close to $\Phi^T \gamma$. This view allows to study various questions inspired by dynamical systems.

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1 Introduction

The theory of permutations offers many exciting structural, extremal and enumerative questions. For example, research centered around the Stanley–Wilf conjecture asks for the number of permutations of a given order avoiding a fixed pattern. What all these problems have in common is that they study static permutations. Following the success of the theory of dense graph limits, Hoppen, Kohayakawa, Moreira, Ráth, and Sampaio [4] developed a theory of permutation limits. The corresponding limit objects are called *permutons*. The theory of permutation limits allowed new results or streamlined proofs in the above areas

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(see e.g. [5, 3]), as well systematic treatment of many properties permutations coming from various random models.

There is another — *dynamic* — line of research of permutations. Much of this line of research is motivated by data structures, and by analysis of sorting algorithms in particular. In the dynamic setting, one studies evolution of sequences $\pi_0, \pi_1, \pi_2, \dots$ of permutations (typically of the same order).

The main contribution of our work is a framework for capturing typical evolutions for a natural class of randomized local algorithms, which we call *flip processes*, using the theory of permutons.

2 Main concepts and results

2.1 Necessary notation

In order to state our results, we need to recall basics of the theory of permutons. All measure below on $[0, 1]^2$ are tacitly assumed to be Borel. We write λ and λ^2 for the Lebesgue measure on \mathbb{R} and on \mathbb{R}^2 , respectively.

A *permuton* γ is a measure on $[0, 1]^2$ with uniform marginals, that is, for each Borel set $Z \subset [0, 1]$ we have that $\gamma([0, 1] \times Z) = \gamma(Z \times [0, 1])$ is the Lebesgue measure of Z . Permutons are an extension of permutations through the concept of permutation representation. Suppose that $\pi \in \text{Sym}(n)$ is a permutation of order n . The *permuton representation* Γ_π of π is a measure defined

$$\Gamma_\pi(X) := n \cdot \lambda^2 \left(X \cap \bigcup_{i=1}^n \left[\frac{i-1}{n}, \frac{i}{n} \right] \times \left[\frac{\pi(i)-1}{n}, \frac{\pi(i)}{n} \right] \right)$$

for each $X \subset [0, 1]^2$. Then Γ_π is indeed a permuton as the fact that each $i \in [n]$ appears exactly once in the domain and exactly once in the range corresponds the uniform marginals on the x-axis and the y-axis, respectively.

We write \mathfrak{P} for the set of all permutons. Given two permutons α and β we define their *rectangular distance* by

$$d_\square(\alpha, \beta) := \sup_{0 \leq x_1 \leq x_2 \leq 1, 0 \leq y_1 \leq y_2 \leq 1} \{ |\alpha([x_1, x_2] \times [y_1, y_2]) - \beta([x_1, x_2] \times [y_1, y_2])| \}. \quad (1)$$

Three permutons which will appear in the text below are the two-dimensional Lebesgue measure λ^2 , the *diagonal permuton* D , defined by $D(Z) = \lambda\{x \in [0, 1] : (x, x) \in Z\}$ for $Z \subset [0, 1]^2$ Borel, and the *antidiagonal permuton* A , defined by $A(Z) = \lambda\{x \in [0, 1] : (x, 1 - x) \in Z\}$.

2.2 Flip processes and permuton trajectories

To motivate our fairly broad class of flip processes, we start with a particular example, a specific ordering procedure. Suppose that π_0 is a permutation of order $n \geq 3$. Then in

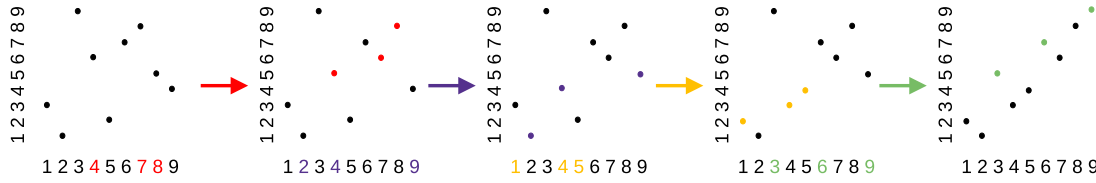


Figure 2.1: Five steps of the ordering process of order 3. The different colours indicate firstly the random choice of the elements and in the subsequent figure the corresponding reordering.

steps $\ell = 1, 2, \dots$ we take a uniform triple of distinct elements of $[n]$, say $i_1 < i_2 < i_3$. We take π_ℓ to be $\pi_{\ell-1}$, except at positions i_1, i_2 , and i_3 and then we shuffle the values $\pi_{\ell-1}(i_1), \pi_{\ell-1}(i_2)$, and $\pi_{\ell-1}(i_3)$ so that they are in the increasing order. We call this process the *ordering process of order 3*. An example is given in Figure 2.1.

Let us now proceed with a general definition. If $\pi \in \text{Sym}(n)$ and $A \in \binom{[n]}{k}$ then *subpermutation of π restricted by A* $\pi|_A$ is a permutation on $[k]$ such that for each $i, j \in [k]$ we have that $\pi|_A(i) < \pi|_A(j)$ if and only if for the i -th smallest element i_A of A and for the j -th smallest element j_A of A we have $\pi(i_A) < \pi(j_A)$. Next, we introduce a notion of *transplanting a subpermutation $\psi \in \text{Sym}(k)$ into π on A* . This is a permutation $\tilde{\pi} \in \text{Sym}(n)$ such for each $i \in [n] \setminus A$ we have $\tilde{\pi}(i) = \pi(i)$ and $\tilde{\pi}|_A = \psi$.

So, the above ordering process can be defined as repeated transplantations of the identity permutation id_3 on randomly selected triples. General flip processes allow the transplanted subpermutation to depend on the sampled restricted subpermutation (and this choice can be a randomized one).

Let $k \in \mathbb{N}$. A *rule* is a stochastic matrix $\mathcal{R} \in [0, 1]^{\text{Sym}(k) \times \text{Sym}(k)}$. Given an initial permutation $\pi_0 \in \text{Sym}(n)$ (where $n \geq k$) the *flip process with rule \mathcal{R}* works as follows. In each step $\ell = 1, 2, \dots$, pick a uniformly random k -tuple $A \in \binom{[n]}{k}$. Then pick a permutation $\psi \in \text{Sym}(k)$ according to the probability distribution given by \mathcal{R} on row $\pi_{\ell-1}|_A$ and transplant it into $\pi_{\ell-1}$ on A . The resulting permutation is π_ℓ . To summarize, a flip process with a given rule is a discrete time-homogeneous Markov process $\pi_0, \pi_1, \pi_2, \dots$.

Suppose that we fix a flip process with a rule \mathcal{R} . The main result of our project states that there is a notion of ‘trajectories’, which are given as a two-variable function $\Phi : \mathfrak{P} \times [0, \infty) \rightarrow \mathfrak{P}$ (in which we write the second coordinate in the superscript, $\Phi^\alpha t$ for $\alpha \in \mathfrak{P}$ and $t \in [0, \infty)$) which predicts typical behaviour of the flip process \mathcal{R} started from any permutation after linearly many steps (with respect to its order), up to a small error in the rectangular distance.

Theorem 1. *For every $k \in \mathbb{N}$ and for every permutation flip process \mathcal{R} of order k , there exists a function $\Phi : \mathfrak{P} \times [0, \infty) \rightarrow \mathfrak{P}$ with the following property. For every $T > 0$, every $n \in \mathbb{N}$ and every $\pi_0 \in \text{Sym}(n)$ we have with probability $1 - o_n(1)$ for the flip process π_0, π_1, \dots with rule \mathcal{R} that $\max \left\{ d_{\square}(\Gamma_{\pi_i}, \Phi^{\frac{i}{n}} \Gamma_{\pi_0}) : i \in (0, Tn] \cap \mathbb{N} \right\} = o_n(1)$.*

That is, Theorem 1 establishes a correspondence between an analytic deterministic

object and a discrete stochastic evolution.

Further, we can prove that the trajectories satisfy the following metric conditions for any $\alpha, \beta \in \mathfrak{P}$ and $s, t \in [0, \infty)$:

$$\exp(-\Theta(t)) d_{\square}(\alpha, \beta) \leq d_{\square}(\Phi^t \alpha, \Phi^t \beta) \leq \exp(\Theta(t)) d_{\square}(\alpha, \beta) \quad \text{and ,} \tag{2}$$

$$d_{\square}(\Phi^s \alpha, \Phi^t \alpha) \leq O(|s - t|) . \tag{3}$$

The upper bound in (2) implies that the evolution of the trajectory depends in a continuous fashion on the initial condition, and (3) says that it also depends in a continuous fashion on time. The lower bound in (2) in particular says that two different trajectories do not ever form a confluence (unless one is a subtrajectory of the other).

2.3 Flip processes as dynamical systems on \mathfrak{P}

Theorem 1 gives a potential of a comprehensive theory of permutation flip processes, richness of which reflects both the combinatorial and the dynamical systems facet of the area. We present several notions that we study. Due to space constraints we state only briefly and informally some of our results accompanying these notions as well as several open questions.

The first is a concept of destination. Suppose that \mathcal{R} is a flip process and $\Phi : \mathfrak{P} \times [0, \infty) \rightarrow \mathfrak{P}$ are its trajectories. If γ is a permuton for which the limit (in the rectangular distance) $\lim_{t \rightarrow \infty} \Phi^t \gamma$ exists, then we call it the *destination of γ* , and write $\text{dest}(\gamma)$. For example, it can be shown that the destination of any permuton in the above ordering process of order 3 is the diagonal permuton D . For most natural flip processes it appears that each permuton has a destination but we are also able to construct a flip process and an initial permuton γ for which $\lim_{t \rightarrow \infty} \Phi^t \gamma$ does not exist. More specifically, we are able to construct a flip process and argue that it contains a periodic trajectory, that is, a permuton γ and time T_0 so that $\Phi^t \gamma = \gamma$ for and only for times t that are multiples of T_0 . It would be interesting to find other wild types of trajectories. For example, does there exist a flip process and an initial permuton α whose trajectory oscillates between the diagonal and the antidiagonal permuton, that is $\liminf_{t \rightarrow \infty} d_{\square}(\Phi^t \alpha, D) = \liminf_{t \rightarrow \infty} d_{\square}(\Phi^t \alpha, A) = 0$?

Destinations are connected with the notion of fixed points. For a flip process whose trajectories are $\Phi : \mathfrak{P} \times [0, \infty) \rightarrow \mathfrak{P}$, we call a permuton $\gamma \in \mathfrak{P}$ a *fixed point* if $\Phi^t \gamma = \gamma$ for all $t \geq 0$. It can be shown that if γ is a destination then it is also a fixed point (and obviously, if it is a fixed point then it is also its own destination). Is it true that every flip process has at least one fixed point?

Next, we explain the concept of origins. Suppose that $t > 0$, and α and β are permutons such that $\beta = \Phi^t \alpha$. In that case we write $\alpha = \Phi^{-t} \beta$. Let $\text{age}(\beta)$ be the supremum of times $t \geq 0$ for which $\Phi^{-t} \beta$ exists as a permuton. It can be shown that if $\text{age}(\beta) < \infty$ then there exists a permuton, denoted by $\text{orig}(\beta)$, for which $\Phi^{\text{age}(\beta)}(\text{orig}(\beta)) = \beta$. We call $\text{orig}(\beta)$ the *origin of β* . Another feature of interest is characterizing graphons with positive age. Indeed, the age of some permutons can be 0 as the example of the antidiagonal permuton A in the ordering process of any order $k \geq 2$ shows. On the positive side, we can show

that if a permuton is absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative is bounded then it is of positive age.

Of course, all the results and questions above would become more tractable if we could, for a any given flip process and any given initial graphon α and any time $t \geq 0$, explicitly compute the permuton $\Phi^t \alpha$. This task however involves solving a difficult system of differential equations and we were able to carry it out only for flip processes of order 2. Last, let us mention the question of the uniqueness of the rule. That is, suppose that we have rules \mathcal{R} and \mathcal{Q} whose respective trajectories are Φ and Ψ . If $\Phi = \Psi$, does it follow that $\mathcal{R} = \mathcal{Q}$?

Related work: Graph flip processes

Our work is similar to, and was in fact inspired by, the theory of flip processes for graphs, recently developed in [2]. Let us summarize that project. We write \mathcal{H}_k for the family of graphs on vertex set $[k]$. A rule \mathcal{R} of a flip process of order k is a stochastic matrix $\mathcal{R} \in [0, 1]^{\mathcal{H}_k \times \mathcal{H}_k}$. For an initial graph G_0 of order $n \geq k$, the flip process with rule \mathcal{R} is a discrete time process $(G_\ell)_{\ell=0}^\infty$ of graphs on the vertex set $[n]$ defined as follows. We get graph $G_{\ell+1}$ by sampling an ordered tuple $\mathbf{v} = (v_1, \dots, v_k)$ of distinct vertices and sample a graph J from distribution $\mathcal{R}_{G_\ell[\mathbf{v}],*}$. We replace $G_\ell[\mathbf{v}]$ by J . The main result of [2] is that there exists trajectories $\Phi \cdot W : [0, +\infty) \rightarrow \mathcal{W}_0$ with properties analogous to (2) and (3) such that if $T > 0$ is a constant and G_0 is an initial graph of large order n then with high probability for each $i \in \mathbb{N}$ with $i \leq Tn^2$ the graphon representation W_i of G_i is close to the trajectory started at the graphon representation W_0 of G_0 at time $\frac{i}{n^2}$ in the cut norm, that is, $\max \left\{ \left\| W_i - \Phi_{\frac{i}{n^2}} W_0 \right\|_{\square} : i \in (0, Tn^2] \cap \mathbb{N} \right\} = o(1)$.

There are substantial similarities between the proofs of the current permutation project and [2] in the overall strategy. In particular, the crucial construction of the trajectories is also based on an idea of a velocity operator (see Section 4). However, there are differences in technical execution of this overall strategy, which are mostly given by combinatorial differences between the cut norm distance for graphons and the rectangular distance for permutons, and of the underlying Banach spaces.¹ Also, families of natural and interesting flip processes seem to be quite different in both cases.²

3 Specific classes of flip processes

We give examples of several classes of flip processes. The purpose of this list is to show richness of scenarios that can be captured by flip processes and to hint to features that can be studied in the future. Many of these processes are counterparts to graph flip processes studied in [1].

¹In the graphon case the Banach space of two-variable L^∞ -functions, and in the permuton case the Banach space of signed measures as we describe in Section 4.

²Interesting graph flip processes are studied in [1].

The *ordering flip process of order k* is given by a rule in which $\mathcal{R}_{\psi, \text{id}_k} = 1$ and $\mathcal{R}_{\psi, \rho} = 0$ for $\rho \neq \text{id}_k$. All trajectories of this flip process converge to the identity permutation. While this is not the only process with this property, intuitively the speed of convergence to the identity permutation (which can be defined analogously to [2, Section 5.12]) is the fastest among all order- k flip processes. The *ignorant flip process* is a process in which the output distribution $\mathcal{R}_{\psi, *}$ does not depend on the input permutation ψ . An example of an ignorant process is the ordering process of any order. In [1, Section 4] ‘ignorant graph processes’ which have an analogous definition are studied. In the graph case, the trajectories can be explicitly described (see [1, Proposition 4.2]), however in the permutation setting this seems to be much more complicated. The *diagonal reversing flip process of order k* is a process designed to swap the order of the permutation in a particular, slow way. It outputs anti-diagonal for input being diagonal and in other cases does not change the permutation. For $k = 2$, it is a fatalistic flip process and the trajectories converge to the anti-diagonal permutation. For $k > 2$ the behaviour is more interesting. In the *complementing flip process of order k* , each input permutation is replaced by its reversal, that is $\mathcal{R}_{\psi, \bar{\psi}} = 1$ where for each $\psi \in \text{Sym}(k)$, $\bar{\psi}$ is defined by $\bar{\psi}(i) := k + 1 - \psi(i)$. All the trajectories converge to the Lebesgue measure λ^2 .

4 Proof of Theorem 1

We sketch the proof of Theorem 1, deferring other results announced in Section 2.3 for the full version of the paper. That is, for a flip process \mathcal{R} of order k , we first need to construct the trajectories $\Phi : \mathfrak{P} \times [0, \infty) \rightarrow \mathfrak{P}$, and then we need to prove that a flip process started with π_0 stays with high probability within a thin sausage around $(\Phi^t \Gamma_{\pi_0})_{t \geq 0}$. Not surprisingly, our construction is tailored with respect to the latter property. More precisely, we work in the Banach space \mathfrak{M} of finite signed Borel measures on $[0, 1]^2$ whose marginals are arbitrary multiples of the 1-dimensional Lebesgue measure, equipped with the distance d_{\square} . We come up with a *velocity operator* $\nabla : \mathfrak{M} \rightarrow \mathfrak{M}$ whose defining formula (5) is explained below. We then require that for $\alpha \in \mathfrak{P}$ and $t \geq 0$ we have the following Banach-space valued equation

$$\frac{d}{dt} \Phi^t \alpha = \nabla \Phi^t \alpha \text{ (differential form), or equivalently } \Phi^t \alpha = \alpha + \int_0^t \nabla \Phi^\tau \alpha \, d\tau \text{ (integral form).} \tag{4}$$

Using certain favorable properties of (4) it can be shown using the theory of Banach-space valued differential equations that it has a unique solution on the entire interval $[0, \infty)$.

We now turn to cooking up the defining formula for ∇ . Recall that in Wormald’s method of differential equation [6], one cooks up real-valued functions whose derivatives are idealizations of expected changes of tracked combinatorial parameters. Our idea is the

same, except our derivatives are \mathfrak{M} -valued. That is, for $\alpha \in \mathfrak{M}$ we set

$$\nabla(\alpha)(Z) = \sum_{\omega \in \text{Sym}(k)} \sum_{i \in [k]} \left(-t(\omega, Z, i; \alpha) + \sum_{\tilde{\omega} \in \text{Sym}(k)} \mathcal{R}_{\omega, \tilde{\omega}} \cdot t(\omega \rightsquigarrow \tilde{\omega}, Z, i; \alpha) \right), \quad (5)$$

for each Borel $Z \subset [0, 1]^2$. Let us explain the motivation behind the quantities $t(\omega, Z, i; \alpha)$ and $t(\omega \rightsquigarrow \tilde{\omega}, Z, i; \alpha)$, which we do under the assumption that α is a permuton.³ The number $t(\omega, Z, i; \alpha)$ is the probability that when sampling⁴ k points from α , we get a permutation ω and further the i -th leftmost point falls in Z . Likewise, $t(\omega \rightsquigarrow \tilde{\omega}, Z, i; \alpha)$ is the probability that when sampling k points from α , we get a permutation ω and further, after swapping the y -coordinates of the sampled points from ω to $\tilde{\omega}$, the i -th leftmost point falls in Z . The corresponding formula (valid again for general $\alpha \in \mathfrak{M}$) for $t(\omega, Z, i; \alpha)$ is (writing $\alpha^{\otimes k}$ for the k -th power of α)

$$t(\omega, Z, i; \alpha) = k! \cdot \alpha^{\otimes k} \left(\left\{ (x_1, y_1, \dots, x_k, y_k) \in [0, 1]^{2k} : x_1 < \dots < x_k \text{ and } (x_i, y_i) \in Z \text{ and for all } \ell, j \in [k] \text{ we have } y_\ell < y_j \text{ if and only if } \omega(\ell) < \omega(j) \right\} \right),$$

and a similar but more complicated formula can be written for $t(\omega \rightsquigarrow \tilde{\omega}, Z, i; \alpha)$.

Having defined the trajectories Φ , we need to prove that a flip process started with a permutation π_0 stays within a thin sausage around $(\Phi^t \Gamma_{\pi_0})_{t \geq 0}$. Actually, we will only prove this for t small. Indeed, if we can prove that with high probability π_{tn} is close to $\Phi^t \Gamma_{\pi_0}$, then we can repeat this argument also starting with permutation $\hat{\pi}_0 := \pi_{tn}$ and get that $\hat{\pi}_{tn} = \pi_{2tn}$ is close to $\Phi^t \Gamma_{\hat{\pi}_0} \approx \Phi^{2t} \Gamma_{\pi_0}$, and more generally, that for any constant $\ell \in \mathbb{N}$, with high probability $\pi_{\ell tn}$ is close to $\Phi^{\ell t} \Gamma_{\pi_0}$, as is needed. (Times between $(\ell - 1)tn$ and ℓtn can be dealt with easily as well.)

So, for $t > 0$ small we use Taylor series approximation of order 1, that is $\Phi^t \Gamma_{\pi_0} \approx \Gamma_{\pi_0} + t \cdot \nabla \Gamma_{\pi_0}$. Recalling that our distance is given by (1), we hence need to prove that with high probability for each $0 \leq x_1 \leq x_2 \leq 1, 0 \leq y_1 \leq y_2 \leq 1$ the quantity

$$|\{i \in [n] \cap [x_1 n, x_2 n] : \pi_{tn}(i) \in [y_1 n, y_2 n]\}| - |\{i \in [n] \cap [x_1 n, x_2 n] : \pi_0(i) \in [y_1 n, y_2 n]\}|$$

is approximately equal to $n \cdot t \cdot \nabla \Gamma_{\pi_0}([x_1, x_2] \times [y_1, y_2])$. This can be proved using concentration inequalities, and making use of the fact that $t(\omega, Z, i; \alpha)$ and $t(\omega \rightsquigarrow \tilde{\omega}, Z, i; \alpha)$ were devised exactly to capture rates of deletions or insertions of points from a permutation in a single step from particular locations.

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³Note that we ultimately, we need to solve (4) in the space of permutons only. However, as is usual in differential equations, to this end we need to work on an open domain. That is, at least some further elements of \mathfrak{M} need to be dealt with.

⁴using the usual procedure of sampling points for a permuton, see Definition 3.2 in [4]

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