

COLOURING COMPLETE MULTIPARTITE AND KNESER-TYPE DIGRAPHS

(EXTENDED ABSTRACT)

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Abstract

The dichromatic number of a digraph D is the smallest k such that D can be partitioned into k acyclic subdigraphs, and the dichromatic number of an undirected graph is the maximum dichromatic number over all its orientations. We present bounds for the dichromatic number of Kneser graphs and Borsuk graphs, and for the list dichromatic number of certain classes of Kneser graphs and complete multipartite graphs. The bounds presented are sharp up to a constant factor. Additionally, we give a directed analogue of Sabidussi's theorem on the chromatic number of graph products.

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We consider graphs and digraphs without loops or multiple edges/arcs. A *proper k -colouring* of a graph $G = (V, E)$ is a mapping $f : V \rightarrow [k] = \{1, \dots, k\}$ such that $f^{-1}(i)$ is an independent set for any $i \in [k]$. The *chromatic number* of G , denoted by $\chi(G)$, is the minimum k for which G has a proper k -colouring. A *proper k -colouring* of a digraph $D = (V, A)$ is a mapping $f : V \rightarrow [k]$ such that $f^{-1}(i)$ is acyclic for any $i \in [k]$, and the *dichromatic number* of D , denoted by $\vec{\chi}(D)$, is the minimum k for which D has a proper k -colouring. Note that this definition generalizes the usual colouring, in the sense that the chromatic number of a graph is equal to the dichromatic number of its corresponding bidirected digraph. The notion was introduced by Neumann-Lara in 1982 [17] and it

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was later rediscovered by Mohar [14]. Since then it has been shown that many classical results hold also in this setting [3, 8, 9, 10]. However, some fundamental questions remain unanswered. The *dichromatic number* of an undirected graph G , denoted by $\vec{\chi}(G)$, is the maximum dichromatic number over all its orientations. Erdős and Neumann-Lara conjectured the following.

Conjecture 1. [5] *For every integer k there exists an integer $r(k)$ such that $\vec{\chi}(G) \geq k$ for any undirected graph G satisfying $\chi(G) \geq r(k)$.*

For instance, $r(1) = 1$ and $r(2) = 3$. But it is already unknown whether $r(3)$ exists. Mohar and Wu [15] managed to prove the fractional analogue of Conjecture 1.

The *Kneser graph* with parameters n, k , denoted by $KG(n, k)$, is the graph with vertex set $\binom{[n]}{k}$ (i.e. the set of subsets of $[n]$ of size k) where two vertices u, v are adjacent if and only if $u \cap v = \emptyset$. It is well-known [7, 12, 13] that $\chi(KG(n, k)) = n - 2k + 2$ for $1 \leq k \leq \frac{n}{2}$, as Kneser conjectured [11, 20]. Providing further evidence for Conjecture 1, Mohar and Wu showed that Kneser graphs with large chromatic number have large dichromatic number.

Theorem 2. [15] *For any positive integers n, k with $1 \leq k \leq \frac{n}{2}$ we have that $\vec{\chi}(KG(n, k)) \geq \left\lfloor \frac{n-2k+2}{8 \log_2 \frac{n}{k}} \right\rfloor$.*

Note that, since $\chi(KG(n, k)) \geq \vec{\chi}(KG(n, k))$, this estimate is sharp up to a constant factor when k is a constant fraction of n . Improving Theorem 2 asymptotically, we show that the dichromatic number of Kneser graphs is of the order of their chromatic number in general.

Theorem 3. *There exists a positive integer n_0 such that, for all $n \geq n_0$ and $2 \leq k \leq \frac{n}{2}$, we have that $\vec{\chi}(KG(n, k)) \geq \lfloor \frac{1}{16} \chi(KG(n, k)) \rfloor$.*

We did not try to optimize the constant $\frac{1}{16}$. The proof of Theorem 3 is based on Greene’s proof of Kneser’s conjecture, but it also relies on Theorem 2 for solving the case of large k . Note that the bound cannot be extended to $k = 1$ (see Theorem 11).

Kneser’s conjecture was an open problem for more than two decades [11, 20]. The famous resolution by Lóvasz [12] was inspired by the analogy between Kneser graphs and Borsuk graphs. Let n be a natural number and $a < 2$ a positive real number. The *Borsuk graph* with parameters $n + 1$ and a , denoted by $BG(n + 1, a)$, is the undirected graph with vertex set $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ where two vertices x, y are adjacent if and only if $\text{dist}_{\mathbb{R}^{n+1}}(x, y) \geq a$. It is known that $\chi(BG(n + 1, a)) \geq n + 2$, which in fact is equivalent to the Borsuk–Ulam theorem [13]. On the other hand, if a is not too small, an $(n + 2)$ -colouring of $BG(n + 1, a)$ can be obtained by projecting the faces of an inscribed $(n + 1)$ -dimensional simplex. Regarding the dichromatic number of Borsuk graphs, we can show the following.

Theorem 4. $\vec{\chi}(BG(n + 1, a)) \geq n + 2$ for any $n \geq 1$.

Next, we look at list colourings. They were introduced by Erdős, Rubin and Taylor [6], and, independently, by Vizing [19]. A *k*-list assignment to a graph $G = (V, E)$ (or to a digraph $D = (V, A)$) is a mapping $L : V \rightarrow \binom{\mathbb{Z}^+}{k}$. A colouring (a mapping) $f : V \rightarrow \mathbb{Z}^+$ is said to be *accepted* by L if $f(v) \in L(v)$ for every $v \in V$. G (resp. D) is *k*-list colourable if every *k*-list assignment accepts a proper colouring. The *list chromatic number* of G (resp. the *list dichromatic number* of D), also called its *choice number*, is the minimum k such that G (resp. D) is *k*-list colourable, and it is denoted by $\chi_\ell(G)$ (resp. $\vec{\chi}_\ell(D)$). Similarly, the *list dichromatic number* of G , denoted by $\vec{\chi}_\ell(G)$, is the maximum list dichromatic number over all orientations of G . Bensmail, Harutyunyan and Le [2] gave a sample of instances where the list dichromatic number of digraphs behaves as its undirected counterpart.

Recently, Bulankina and Kupavskii [4] studied the list chromatic number of Kneser graphs. They proved the following two results.

Theorem 5. [4] *For any positive integers n, k with $1 \leq k \leq \frac{n}{2}$ we have that $\chi_\ell(KG(n, k)) \leq n \ln \frac{n}{k} + n$.*

Theorem 6. [4] *Let $s \geq 3$ be an integer. If n is sufficiently large and $3 \leq k \leq n^{1/2-1/s}$, then $\chi_\ell(KG(n, k)) \geq \frac{1}{2s^2}n \ln n$. For $k = 2$, we have that $\chi_\ell(KG(n, k)) \geq \frac{1}{32}n \ln n$ for sufficiently large n .*

However, good bounds for larger k are still unknown. Using the arguments of Bulankina and Kupavskii, as well as ideas from [15], we can prove the directed version of Theorem 6.

Theorem 7. *For every $\varepsilon \in \mathbb{R}^+$ there exists a constant $c_\varepsilon \in \mathbb{R}^+$ such that $\vec{\chi}_\ell(KG(n, k)) \geq c_\varepsilon n \ln n$ for all $n \geq 2k$ with $2 \leq k \leq n^{1/2-\varepsilon}$.*

Dense Kneser graphs have some similarities with complete multipartite graphs. Denote by K_{m^*r} the complete r -partite graph with m vertices on each part. Alon determined, up to a constant factor, the list chromatic number of K_{m^*r} , answering a question of Erdős, Rubin and Taylor [6].

Theorem 8. [1] *There exist two positive constants c_1 and c_2 such that for every $m \geq 2$ and for every $r \geq 2$*

$$c_1 r \ln m \leq \chi_\ell(K_{m^*r}) \leq c_2 r \ln m.$$

His proof can be adapted to find an analogous bound for the list dichromatic number of K_{m^*r} when m is not too small.

Theorem 9. *For every $\rho > 3$, there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that if $r \geq 2$ and $m \geq \ln^\rho r$ then*

$$C_1 r \ln m \leq \vec{\chi}_\ell(K_{m^*r}) \leq C_2 r \ln m.$$

In what follows we present a proof of Theorem 9. The following probabilistic result will be required.

Theorem 10. (Simple Concentration Bound, [16]) *Let X be a random variable determined by n independent trials, and satisfying the property that changing the outcome of any single trial can affect X by at most c . Then*

$$\mathbb{P}(|X - \mathbb{E}X| > t) \leq 2e^{-\frac{t^2}{2c^2n}}.$$

Proof of Theorem 9. The upper bound is implied by Theorem 8. We may assume that m is large enough. Let V_1, \dots, V_r be the parts of K_{m*r} .

Claim. There is a constant c and an orientation D of K_{m*r} such that, if $\ell \geq c \ln(rm)$, then

- (i) each subgraph of K_{m*r} isomorphic to K_ℓ has a directed cycle in D ;
- (ii) for each $U_i \subseteq V_i$ and $U_j \subseteq V_j$ with $|U_i| = |U_j| = \ell$ and $i \neq j$, $D[U_i \cup U_j]$ has a directed cycle.

Proof. Orient the edges of K_{m*r} at random, independently and with probability $\frac{1}{2}$. Let E, E' be the events that (i), (ii) hold, respectively. Put $\ell = \lceil c \ln(rm) \rceil$. There are $\binom{r}{\ell} m^\ell$ copies of K_ℓ in K_{m*r} , and $\binom{r}{2} \binom{m}{\ell}^2$ subgraphs of the form $K_{m*r}[U_i \cup U_j]$. Furthermore, K_ℓ (resp. $K_{m*r}[U_i \cup U_j]$) has $2^{\frac{\ell(\ell-1)}{2}}$ orientations (resp. 2^{ℓ^2}), among which $\ell!$ (resp. at most $(2\ell)!$) are acyclic. Therefore,

$$\mathbb{P}(E^c) \leq \binom{r}{\ell} m^\ell \ell! 2^{-\frac{\ell(\ell-1)}{2}} \leq (rm 2^{-\frac{\ell-1}{2}})^\ell \leq (e^{\frac{\ell}{c}} 2^{-\frac{\ell-1}{2}})^\ell < \frac{1}{2} \text{ and}$$

$$\mathbb{P}(E'^c) \leq \binom{r}{2} \binom{m}{\ell}^2 (2\ell)! 2^{-\ell^2} \leq (2er^2 m^2 2^{-\ell})^\ell \leq (e^{\frac{2\ell}{c} + 1} 2^{-\ell + 1})^\ell < \frac{1}{2}$$

if c is large enough. Hence $\mathbb{P}(E \cap E') > 0$ for some c . □

Let $k = \lfloor Cr \ln m \rfloor$, where $0 < C \leq 1$ is a constant for now unspecified. We start by showing that there exists an assignment of k -lists from a palette \mathcal{C} of $\lfloor r \ln m \rfloor$ colours such that, for any given set $A \subseteq \mathcal{C}$ of at most $\frac{4}{3} \ln m$ colours, each part has at least $\frac{1}{2} m^{1-\delta}$ vertices that avoid the colours from A on their lists, where $\delta = 2C \ln 5$.

Assign to each vertex v of D a random k -list $L(v)$ chosen independently and uniformly among the $\binom{|\mathcal{C}|}{k}$ possible k -lists. Given $i \in [r]$ and $A \subseteq \mathcal{C}$, consider the random variable $X_{i,A} = |\{v \in V_i \mid L(v) \cap A = \emptyset\}|$. Note that there are exactly $\binom{|\mathcal{C}| - |A|}{k}$ k -lists avoiding the colours in A . Devoting ourselves to the case $|A| = \lfloor \frac{4}{3} \ln m \rfloor$, we have that

$$\begin{aligned} \mathbb{E}X_{i,A} &= m \frac{\binom{|\mathcal{C}| - |A|}{k}}{\binom{|\mathcal{C}|}{k}} \geq m \left(\frac{|\mathcal{C}| - |A| - k}{|\mathcal{C}| - k} \right)^k = m \left(1 - \frac{|A|}{|\mathcal{C}| - k} \right)^k \\ &\geq m \left(1 - \frac{\frac{4}{3} \ln m}{(1 - C)r \ln m - 1} \right)^{Cr \ln m} \geq m \left(1 - \frac{4}{5} \right)^{2C \ln m} = m^{1-\delta} \end{aligned}$$

if m is large enough and C is not too large. By the Simple Concentration Bound (Theorem 10),

$$\mathbb{P}(X_{i,A} < \frac{1}{2}m^{1-\delta}) \leq \mathbb{P}(|X_{i,A} - \mathbb{E}X_{i,A}| > \frac{1}{2}m^{1-\delta}) \leq 2e^{-\frac{1}{8}m^{1-2\delta}}.$$

Let E be the event that $X_{i,A} < \frac{1}{2}m^{1-\delta}$ for some $i \in [r]$ and $A \subseteq \mathcal{C}$ with $|A| \leq \frac{4}{3} \ln m$. We have that

$$\begin{aligned} \mathbb{P}(E) &\leq r \binom{|\mathcal{C}|}{\lfloor \frac{4}{3} \ln m \rfloor} 2e^{-\frac{1}{8}m^{1-2\delta}} \leq r \left(e^{\frac{r \ln m}{\lfloor \frac{4}{3} \ln m \rfloor}} \right)^{\lfloor \frac{4}{3} \ln m \rfloor} 2e^{-\frac{1}{8}m^{1-2\delta}} \\ &\leq r(er)^{\frac{4}{3} \ln m} 2e^{-\frac{1}{8}m^{1-2\delta}} \leq 2e^{5 \ln r \ln m - \frac{1}{8}m^{1-2\delta}} \leq 2e^{5m^{\frac{1}{\rho}} \ln m - \frac{1}{8}m^{1-2\delta}} \end{aligned}$$

if m is large enough. Consequently, if $\delta < \frac{1}{2}(1 - \frac{1}{\rho})$ and m is large enough, there exists a list assignment L' satisfying the desired property. This is the assignment that we are going to use.

Now let f be a proper colouring of D . We claim that there exists a set of indices $I \subseteq [r]$ of size at least $\frac{3r}{4}$ such that $|f(V_i)| \leq 4c \ln^2(rm)$ for each $i \in I$. Indeed, if more than $\frac{r}{4}$ parts are coloured with more than $4c \ln^2(rm)$ colours each, then one of the colours appears on more than $\frac{cr \ln^2(rm)}{|\mathcal{C}|} \geq c \frac{\ln^2(rm)}{\ln m} \geq c \ln(rm)$ parts. By the choice of D , f is not proper, a contradiction.

For each $i \in [r]$ define the set $A_i = \{\gamma \in \mathcal{C} \mid |V_i \cap f^{-1}(\gamma)| \geq c \ln(rm)\}$. We claim that if f is acceptable then $|A_i| > \frac{4}{3} \ln m$ for every $i \in I$. Indeed, otherwise, by the choice of the lists, at least $\frac{1}{2}m^{1-\delta}$ vertices of V_i have been coloured with colours not from A_i . Thus one of these colours is used at least

$$\frac{\frac{1}{2}m^{1-\delta}}{4c \ln^2(rm)} \leq c \ln(rm)$$

times on V_i . If m is large enough, this implies that

$$m^{1-\delta} \leq 8c^2 \ln^3(rm) \leq 8c^2(m^{\frac{1}{\rho}} + \ln m)^3 \leq 9c^2m^{\frac{3}{\rho}}.$$

If we further assume that $\delta < 1 - \frac{3}{\rho}$, we get a contradiction when m is large. Therefore $|A_i| > \frac{4}{3} \ln m$ for every $i \in I$.

Now, by the choice of D , the sets A_1, \dots, A_r are mutually disjoint. But then

$$|\mathcal{C}| \geq \sum_{i=1}^r |A_i| \geq \sum_{i \in I} |A_i| > \frac{4}{3}|I| \ln m \geq r \ln m \geq |\mathcal{C}|.$$

This contradiction shows that there is no acceptable proper colouring for the k -list assignment L' . □

We do not know what happens with other values of m, r . What is clear is that the bound of Theorem 9 is not valid in general. Indeed, if $m \leq \ln r$ then Theorem 11 implies that $\vec{\chi}(K_{m^*r}) \leq \vec{\chi}(K_{mr}) \leq cr$ for some constant c .

Theorem 11. [2] *Let T be a tournament of order n . Then $\vec{\chi}_\ell(T) \leq \frac{n}{\log_2 n}(1 + o(1))$.*

Some of our proofs rely on graph products. Let G, H be graphs (resp. digraphs). The *Cartesian product* of G and H is the graph (resp. digraph) $G \square H$ with vertex set $V(G) \times V(H)$ where there is an edge between (u, x) and (v, y) (resp. an arc from (u, x) to (v, y)) if and only if either $u = v$ and $\{x, y\} \in E(H)$ (resp. and $(x, y) \in A(H)$), or $x = y$ and $\{u, v\} \in E(G)$ (resp. and $(u, v) \in A(G)$). A well-known theorem of Sabidussi [18] states that for any two graphs G and H the chromatic number of its product is $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. His proof can be adapted to show an analogous result for digraphs.

Theorem 12. *Let G and H be digraphs. Then $\vec{\chi}(G \square H) = \max\{\vec{\chi}(G), \vec{\chi}(H)\}$.*

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