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3-uniform linear hypergraphs without a long Berge path

(Extended abstract)

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Abstract

Extensions of the Erdős-Gallai theorem for general hypergraphs are well studied. In this work, we prove the extension of the Erdős-Gallai theorem for linear hypergraphs. In particular, we show that the number of hyperedges in an n-vertex 3-uniform linear hypergraph, without a Berge path of length k as a subgraph is at most $\frac{(k-1)}{6}n$ for $k \geq 4$. This is an extended abstract for EUROCOMB23 of the manuscript arXiv:2211.16184.

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1 Introduction

Finding the maximum number of edges in a graph with fixed order not containing another graph as a subgraph is a central problem in extremal combinatorics. This work considers problems where a path of fixed length is forbidden. This problem is well understood for graphs and r-uniform hypergraphs. The Erdős–Gallai theorem states that a graph of order n containing no path of length k as a subgraph contains at most $\frac{k-1}{2}n$ edges. This bound is sharp for infinitely many n. In particular, equality holds if and only if n is a multiple of k and the graph is isomorphic to the union of $\frac{n}{k}$ cliques of size k. This theorem was extended to r-uniform hypergraphs by Győri, Katona and Lemons [11]. In order to state their result, we will introduce the necessary definitions.

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For an integer r, a hypergraph $\mathcal{H}$ is r-uniform if it is a family of r-element sets of finite family $V(\mathcal{H})$. We will use the following extension of this definition. For a set of integers R, a hypergraph $\mathcal{H}$ is R-uniform if it is a family of sets of the finite family $V(\mathcal{H})$, such that the sizes of the sets are elements of R. Paths in hypergraphs can be defined in a number of ways. In this paper, we follow the definition of Berge [2]. A Berge path of length k in a hypergraph $\mathcal{H}$ is an alternating sequence $v_1, h_1, v_2, \dots, h_k, v_{k+1}$ of distinct vertices and hyperedges such that $\{v_i, v_{i+1}\} \subseteq h_i$ for all $i \in [k]$. A Berge cycle of length k is also defined similarly. The vertices v_i, $i \in [k+1]$, are defining vertices of the Berge path and the hyperedges h_i, $i \in [k]$, are defining hyperedges of the Berge path.

Theorem (Győri, Katona and Lemons [11])**.** *Let $\mathcal{H}$ be an n-vertex r-uniform hypergraph containing no Berge path of length k as a subgraph. Then if $r \geq k > 2$ then the number of hyperedges of $\mathcal{H}$ is at most $\frac{k-1}{r+1}n$. If $k > r+1 > 2$ then the number of hyperedges of $\mathcal{H}$ is at most $\frac{\binom{k}{r}}{k}n$.*

The remaining case $k = r+1$ was settled later in [3], the bound matches with the bound in Theorem 1 for $k > r+1$ case. Forbidden path problems for connected graphs and hypergraphs including their stability versions are well studied, we refer interested readers to [16, 1, 13, 6, 15, 8, 7, 9]. Uniform hypergraphs with bounded circumference was studied in [5, 12] and references therein.

Here we introduce some necessary technical definitions. For a hypergraph $\mathcal{H}$ let $E(\mathcal{H})$ be the hyperedge set and $V(\mathcal{H})$ be the vertex set, we denote their sizes by $e(\mathcal{H})$ and $v(\mathcal{H})$ accordingly. The hypergraph $\mathcal{H}$ is linear if for any two distinct hyperedges h_1, h_2 we have $|h_1 \cap h_2| \leq 1$. For a vertex set V, $V \subseteq V(\mathcal{H})$, we define another hypergraph $\mathcal{H}_V$. Where $V(\mathcal{H}_V) = V$ and $E(\mathcal{H}_V) = \{h \setminus V : h \in E(\mathcal{H}), |h \setminus V| \geq 2\}$. Note that if $\mathcal{H}$ is $\{2,3\}$-uniform linear hypergraph then $\mathcal{H}_V$ is $\{2,3\}$-uniform linear hypergraph also. The induced hypergraph on the vertex set V is denoted by $\mathcal{H}[V]$. For a hypergraph $\mathcal{H}$ we denote two-shadow of $\mathcal{H}$ by $\partial\mathcal{H}$. It is a graph on the same vertex set as $\mathcal{H}$ and the set of edges is $\{\{u,v\} : \{u,v\} \subseteq h \in E(\mathcal{H})\}$. The degree of a vertex v in a hypergraph $\mathcal{H}$ is the number of hyperedges incident to the vertex v and is denoted by $d_{\mathcal{H}}(v)$. The minimum degree of a vertex in a hypergraph $\mathcal{H}$ is denoted by $\delta_{\mathcal{H}}(v)$. The circumference of $\mathcal{H}$ is the length of the longest Berge cycle in a hypergraph $\mathcal{H}$ and is denoted by $c(\mathcal{H})$. The neighborhood of a vertex v in a hypergraph $\mathcal{H}$ is denoted by $N_{\mathcal{H}}(v)$. For a hypergraph $\mathcal{H}$ and sub-hypergraph $\mathcal{H}'$ we denote the hypergraph on the same vertex set as $\mathcal{H}$ and hyperedge set $E(\mathcal{H}) \setminus E(\mathcal{H}')$ by $\mathcal{H} \setminus \mathcal{H}'$.

2 Main results

Recently Gyárfás, Ruszinkó, and Sárközy [10] initiated the study of three uniform linear hypergraphs not containing a linear path, a matching, and a small tree. In particular, they proved that the maximum number of hyperedges in an n vertex three uniform linear hypergraph not containing a linear path of k edges is $1.5nk$. In this paper, we prove the extension of Erdős-Gallai theorem for linear 3-uniform hypergraphs but instead of forbidding linear paths, we forbid Berge paths.

Theorem 1. *Let $\mathcal{H}$ be an n vertex 3-uniform linear hypergraph, containing no Berge path of length $k \geq 4$. Then the number of hyperedges in $\mathcal{H}$ is at most $\frac{k-1}{6}n$.*

Note that the upper bound is sharp for infinitely many k and n. In particular for all k for which there exists a Steiner Triple System (a 3-uniform hypergraph that every pair of vertices is covered by precisely one hyperedge) and n multiple of k, there exists an n-vertex 3-uniform linear hypergraph $\mathcal{H}$, containing no Berge path of length k with $\frac{k-1}{6}n$ hyperedges. Where $\mathcal{H}$ is the disjoint union of $\frac{n}{k}$ copies of k-vertex Steiner Triple Systems.

In order to prove Theorem 1 with induction for k, we need a stronger and more general statement of the theorem.

Theorem 2. *Let $\mathcal{H}$ be an n vertex $\{2,3\}$-uniform linear hypergraph, containing no Berge path of length $k \geq 4$. Then the number of edges in $\partial\mathcal{H}$ is at most $\frac{k-1}{2}n$.*

Note that Theorem 1 is a direct corollary of Theorem 2. The following remark shows that the condition $k \leq 4$ in Theorem 2 is necessary since for $k < 4$ we have different bounds.

Remark. *Let $\mathcal{H}$ be an n vertex linear $\{2,3\}$-uniform hypergraph, containing no Berge path of length k.*

- *If $k = 1$ then $e(\partial\mathcal{H}) = 0$;*
- *If $k = 2$ then $e(\partial\mathcal{H}) \leq v(\mathcal{H})$; The upper-bound is sharp and the equality is achieved if and only if is $v(\mathcal{H})$ multiple of 3 and $\mathcal{H}$ is $\frac{v(\mathcal{H})}{3}$ independent hyperedges of size three.*
- *If $k = 3$ then $e(\partial\mathcal{H}) \leq 3\frac{v(\mathcal{H})-1}{2}$. The upper-bound is sharp and the equality is achieved if and only if $v(\mathcal{H})$ is odd and $\mathcal{H}$ is $\frac{v(\mathcal{H})-1}{2}$ hyperedges of size three sharing the same vertex for every $n \geq 3$.*

We find it challenging to obtain the precise bound for the problem initiated by Gyárfása, Ruszinkó, and Sárközy [10]. Consequently, we would like to put forth a natural conjecture.

Conjecture 3. *Let $\mathcal{H}$ be an n vertex 3-uniform linear hypergraph, containing no linear path of length $k \geq 5$. Then the number of hyperedges in $\mathcal{H}$ is at most $\frac{2k-1}{6}n$.*

Note that, this bound is sharp for infinitely many pairs of n and k. In particular for every k such that there exists a Steiner Triple System on $2k$ vertices and for every n multiple of $2k$. The hypergraph containing $\frac{n}{2k}$ copies of a Steiner Triple System on $2k$ vertices achieves the desired bound.

3 Proof of Theorem 2

For the full proof see manuscript [14].

We prove Theorem 2 by induction on k. At first, we consider the base case when $k = 4$. We may assume $\mathcal{H}$ is a connected hypergraph since the upper bound is linear for n and the additive constant is 0. If $\mathcal{H}$ is Berge cycle free then $e(\partial\mathcal{H}) \leq \frac{3(n-1)}{2}$ (the upper-bound is attained by hyperedges of size three sharing a fixed vertex). If $\mathcal{H}$ contains a Berge cycle it must be a Berge cycle of length 3 or 4 since it is a linear hypergraph. If $\mathcal{H}$ contains Berge cycle of length 4 then by connectivity $v(\mathcal{H}) \leq 4$, hence $e(\mathcal{H}) \leq \binom{4}{2} = \frac{3n}{2}$. If $\mathcal{H}$ contains a cycle of length 3, we denote it by C_3. Cycle C_3 is a linear cycle since $\mathcal{H}$ is a linear hypergraph. If all of the hyperedges of C_3 are size three then by the connectivity of $\mathcal{H}$ we have $\mathcal{H} = C_3$ and $e(\partial\mathcal{H}) = 9 = \frac{3n}{2}$. If two of the hyperedges are size three then by the connectivity of $\mathcal{H}$ we have $\mathcal{H} = C_3$ and $e(\partial\mathcal{H}) = 7 < \frac{3n}{2}$. If at most one hyperedge is size three then we have $e(\partial\mathcal{H}) \leq \frac{3n}{2}$. So the base case $k = 4$ is done.

Let $\mathcal{H}$ be an n-vertex linear $\{2,3\}$-uniform hypergraph containing no Berge path of length k for some integer $k > 4$. Suppose by way of contradiction that $e(\partial\mathcal{H}) > \frac{n(k-1)}{2}$. Without loss of generality, we may assume n is minimal, in particular, we assume all linear $\{2,3\}$-uniform hypergraphs containing no Berge path of length k with n' vertices, $n' < n$, contain at most $\frac{n(k-1)}{2}$ edges in the shadow. Note that from the minimality of n we have the hypergraph $\mathcal{H}$ is connected. Even more, for each vertex v, $\mathcal{H}_{V(\mathcal{H})\setminus\{v\}}$ contains no Berge path of length k, thus from the minimality of n we have $d_{\partial\mathcal{H}}(v) > \frac{k-1}{2}$. Hence we have $\delta_{\partial\mathcal{H}}(v) \geq \lceil \frac{k}{2} \rceil$. Note that since $e(\partial\mathcal{H}) > \frac{n(k-1)}{2}$ the longest path of $\mathcal{H}$ is length $k-1$ by the induction hypothesis.

We omit the proof of the following Claims.

Claim 4. $c(\mathcal{H}) \geq \lceil \frac{k+1}{2} \rceil$.

Let $\mathcal{C}_\ell := v_1, h_1, v_2, h_2, \ldots h_{\ell-1}, v_\ell, h_\ell, v_1$ be a longest Berge cycle of $\mathcal{H}$. Some $\mathcal{C}_\ell$ defining hyperedges h_i are size three, let us denote the third vertex by x_i, that is $h_i = \{v_i, v_{i+1}, x_i\}$ for hyperedges of size three. From Claim 4 we have $\ell \geq \lceil \frac{k+1}{2} \rceil$. Let us denote the hypergraph $\mathcal{H}_{V(\mathcal{H})\setminus\{v_i : i\in[\ell]\}}$ by $\mathcal{H}'$.

Claim 5. *The hypergraph $\mathcal{H}'$ is $\mathcal{BP}_{k-\ell}$-free.*

If $k - \ell \geq 4$ then by Claim 5 and induction hypothesis for hypergraph $\mathcal{H}'$ we have

$$e(\partial\mathcal{H}') \leq \frac{(n-\ell)(k-\ell-1)}{2}. \tag{1}$$

For a vertex $u \in V(\mathcal{H}')$ we define the set $S(u) := N_{\mathcal{H}\setminus C_\ell}(u) \cap V(C_\ell)$, $L(u) := \{v_i : u = x_i\}$ and $R(u) := \{v_{i+1} : u = x_i\}$. For a vertex set S such that $S \subseteq V(C_\ell)$ let S^+ be a set S shifted right, in particular $S^+ := \{v_i : v_{i-1} \in S\}$, the indices are taken module ℓ. Similarly we definite S^-, in particular S^- is a set for which $S = (S^-)^+$. Naturally we denote the set $(S^-)^-$ with S^{--} and the set $(S^+)^+$ with S^{++}. Note that $L(u)^+ = R(u)$, thus the size of $L(u)$ and $R(u)$ are the same.

In what follows we are going to estimate the number of edges in $\partial\mathcal{H}$, in the following way

$$e(\partial\mathcal{H}) = e(\partial\mathcal{H}_{V(C_\ell)}) + e_{\partial\mathcal{H}}(V(C_\ell), V(\mathcal{H}')) + e(\partial\mathcal{H}'). \tag{2}$$

Noting that $e_G(A, B)$ denotes the number of edges between vertex set A and B in G. In most cases, we will use a naive upper bound for $e(\partial\mathcal{H}_{V(C_\ell)}) \leq \binom{\ell}{2}$. For $k - \ell \geq 4$, we estimate $e(\partial\mathcal{H}')$ by the induction hypotheses as in Equation 1. We estimate the number of edges from $V(\mathcal{H}')$ to $V(C_\ell)$, for each vertex $u \in V(\mathcal{H}')$ in $\partial\mathcal{H}$. In particular the number of adjacent vertices to u is $|L(u)| + |R(u)| + |S(u)|$. Since each defining hyperedge of C_ℓ provides at most two edges crossing between the vertices $V(\mathcal{H}')$ and $V(C_\ell)$ we have a naive upper bound for $e_{\partial\mathcal{H}}(V(C_\ell), V(\mathcal{H}'))$ which is enough for most of the cases.

$$e_{\partial\mathcal{H}}(V(C_\ell), V(\mathcal{H}')) \leq 2\ell + \sum_{u \in V(\mathcal{H}')} |S(u)|. \tag{3}$$

Since C_ℓ is a longest Berge cycle of $\mathcal{H}$ we are able to get an upper bound for $|S(u)|$ from the following claim.

Claim 6. *For a vertex $u \in V(\mathcal{H}')$ we have $(S(u) \cup L(u)) \cap S(u)^- = \emptyset$.*

Note that if a vertex $v_i \in S(u)$ then $v_{i+1} \notin S(u)$ from Claim 6. Thus we have $|S(u)| \leq \frac{\ell}{2}$ for each vertex u of $\mathcal{H}'$. Therefore $e_{\partial\mathcal{H}}(V(C_\ell), V(\mathcal{H}')) \leq 2\ell + \frac{\ell}{2}(n - \ell)$ from Equation 3. If $k - \ell \geq 4$ then by Equation 2 and 1 we have a contradiction

$$e(\partial\mathcal{H}) \leq \binom{\ell}{2} + 2\ell + \frac{\ell(n-\ell)}{2} + \frac{(n-\ell)(k-\ell-1)}{2} = \frac{n(k-1)}{2} + \frac{\ell}{2}(\ell + 4 - k) \leq \frac{n(k-1)}{2}.$$

We study the rest of the possible values of ℓ separately, $\ell \in \{k-3, k-2, k-1, k\}$. Let x be the number of defining hyperedges of C_ℓ incident to a vertex of $\mathcal{H}'$. Note that $0 \leq x \leq \ell$.

If $\ell = k$ then $\mathcal{C}_\ell = \mathcal{H}$ otherwise we have a Berge path of length k in $\mathcal{H}$ by the connectivity of $\mathcal{H}$. Thus we have $n = k = \ell$ and

$$e(\partial\mathcal{H}) \leq \binom{\ell}{2} = \frac{n(k-1)}{2}.$$

If $\ell = k - 1$ then $\mathcal{H}'$ contains no hyperedge by Claim 5. Since $\mathcal{H}$ does not contain a Berge path of length k, if a hyperedge h_i adjacent to a vertex from $V(\mathcal{H}')$, then neither v_i nor v_{i+1} is a vertex of $S(u)$, for all $u \in V(\mathcal{H})$. In particular for $u, u' \in V(\mathcal{H}')$ we have $L(u) \cap (S(u))^- = \emptyset$. By this observation and Claim 6 every vertex of $V(\mathcal{H}')$ is adjacent to at most $\frac{k-1-x}{2}$ vertices of C_ℓ with a non-defining hyperedge, that is $|S(u)| \leq \frac{k-1-x}{2}$. Thus by Equation 2 we have

$$e(\partial\mathcal{H}) \leq \binom{k-1}{2} + 2x + \frac{k-1-x}{2}(n - (k-1)).$$

Hence if $n \geq k+2$ or $n = k+1$ and $x \leq \frac{k-1}{2}$ then we have $e(\partial\mathcal{H}) \leq \frac{n(k-1)}{2}$, since $x \leq k-1$. As $e(\partial\mathcal{H}) > \frac{n(k-1)}{2}$ we have $n \geq k+1$.

If $n = k+1$ and $x > \frac{k-1}{2}$ then there are two C_ℓ non-defining hyperedges h_i and h_{i+1} such that $\{x_i, x_{i+1}\} = V(\mathcal{H}')$. Since $\mathcal{H}$ does not contain a Berge path of length k, if a defining vertex of C_ℓ is incident to both vertices of $\mathcal{H}'$, either both incidences are from a defining hyperedge or both incidences are from a non-defining hyperedge. If v_j is incident to both vertices of $\mathcal{H}'$ with h_{j-1} and h_j such that $j \neq i-1, i, i+1$ then v_j is not incident to v_{i+1}. Otherwise, if there is a hyperedge f' incident to v_j and v_{j+1}, then it is a non-defining hyperedge and the following is a Berge path or a Berge cycle of length k,

$$x_{i+1}, h_{i+1}, v_{i+2}, \dots, v_j, f', v_{i+1}, h_i, v_i, \dots, v_{j+1}, h_j, x_j.$$

If a vertex v_j is adjacent to x_1 or x_2 with a non-defining hyperedge then v_{j+1} is not adjacent to a vertex from $\{x_1, x_2\}$. Thus for each vertex $v_j \in V(C_\ell)$, $j \notin \{i-1, i, i+1\}$, either there is at most one vertex from $V(\mathcal{H}')$ adjacent to it, or if there are two then $v_j v_{i+1}$ is not an edge of $\partial\mathcal{H}$ or v_{j+1} is not adjacent to any vertex of $V(\mathcal{H}')$. Note that if there is a defining hyperedge of C_ℓ not incident to a vertex of $\mathcal{H}'$ then we may choose i such that $i-1$ has exactly one neighbor in $V(\mathcal{H}')$. If all defining hyperedges of C_ℓ are incident to a vertex of $\mathcal{H}'$ then we may choose any i from $[k-1]$. Thus we have a contradiction from Equation 2

$$e(\partial\mathcal{H}) \leq \binom{k-1}{2} + k - 1 + 2 \leq \frac{n(k-1)}{2}.$$

The proof of remaining cases $\ell = k-2$ and $\ell = k-3$ involves more structural study and can be seen in the original manuscript.

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