

# THE MAXIMUM NUMBER OF COPIES OF AN EVEN CYCLE IN A PLANAR GRAPHS

(EXTENDED ABSTRACT)

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## Abstract

We resolve a conjecture of Cox and Martin by determining asymptotically for every  $k \geq 2$  the maximum number of copies of  $C_{2k}$  in an  $n$ -vertex planar graph.

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## 1 Introduction

A fundamental problem in extremal combinatorics is maximizing the number of occurrences of subgraphs of a certain type among all graphs from a given class. In the case of  $n$ -vertex planar graphs, Hakimi and Schmeichel [8] determined the maximum possible number of cycles of length 3 and 4 exactly and showed that for any  $k \geq 3$ , the maximum number of

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$k$ -cycles is  $\Theta(n^{\lfloor k/2 \rfloor})$ . Moreover, they proposed a conjecture for the maximum number of 5-cycles in an  $n$ -vertex planar graph which was verified much later by Győri *et al.* in [6]. The maximum number of 6-cycles and 8-cycles was settled asymptotically by Cox and Martin in [3], and later the same authors [4] also determined the maximum number of 10-cycles and 12-cycles asymptotically.

Following the work of Hakimi and Schmeichel [8], Alon and Caro [1] considered the general problem of maximizing copies of a given graph  $H$  among  $n$ -vertex planar graphs. Wormald [11] and later independently Eppstein [5] showed that for 3-connected  $H$ , the maximum number of copies of  $H$  is  $\Theta(n)$ . The order of magnitude in the case when  $H$  is a tree was determined in [7], and the order of magnitude for an arbitrary graph was settled by Huynh, Joret and Wood [9]. Note that by Kuratowski's theorem [10] such problems can be thought of as generalized Turán problems where we maximize the number of copies of the graph  $H$  while forbidding all subdivisions of  $K_5$  and  $K_{3,3}$ .

Given that the order of magnitude of the maximum number of copies of any graph  $H$  in an  $n$ -vertex planar graph is determined, it is natural to look for sharp asymptotic results. While in recent times a number of results have been obtained about the asymptotic number of  $H$ -copies in several specific cases, less is known for general classes of graphs. Cox and Martin [3] introduced some general tools for studying such problems and conjectured that in the case of an even cycle  $C_{2k}$  with  $k \geq 3$ , the maximum number of copies is asymptotically  $n^k/k^k$ . We confirm their conjecture.

**Theorem 1.** *For every  $k \geq 3$ , the maximum number of copies of  $C_{2k}$  in an  $n$ -vertex planar graph is*

$$\frac{n^k}{k^k} + o(n^k).$$

A construction containing this number of copies of  $C_{2k}$  is obtained by taking a  $C_{2k}$  and replacing every second vertex by an independent set of approximately  $n/k$  vertices, each with the same neighborhood as the original vertex. Cox and Martin [3] proved that a weaker upper bound of  $\frac{n^k}{k!} + o(n^k)$  holds for the number of copies of  $C_{2k}$  and introduced a general method for (asymptotically) maximizing the number of copies of a large variety of graphs in a planar graph. We will discuss this method in Section 2 and present another conjecture of Cox and Martin which implies Theorem 1. In Section 3, we prove this stronger conjecture (Theorem 2). We have learned that Asaf Cohen Antonir and Asaf Shapira have independently obtained a bound within a factor of  $e$  of the optimal bound attained in Theorem 2.

## 2 Reduction lemma of Cox and Martin

For a positive integer  $n$  we will consider functions  $w : E(K_n) \rightarrow \mathbb{R}$  satisfying the conditions:

1. For all  $e \in E(K_n)$ ,  $w(e) \geq 0$ ,
2.  $\sum_{e \in E(K_n)} w(e) = 1$ .

For a subgraph  $H'$  of  $K_n$  and a function  $w$  satisfying Conditions 1 and 2, let

$$p_w(H') := \prod_{e \in E(H')} w(e).$$

Also for a fixed graph  $H$  and  $w$  satisfying Conditions 1 and 2 let

$$\beta(w, H) := \sum_{H \cong H' \subseteq K_n} p_w(H').$$

For simplicity of notation, we will often omit statements about isomorphism in the sums. Cox and Martin proved several reduction lemmas for pairs of graphs  $H$  and  $K$ , in which an optimization problem involving  $\beta(w, K)$  implies a corresponding upper bound on the maximum number of copies of the graph  $H$  among  $n$ -vertex planar graphs. We state the reduction lemma which Cox and Martin proved for cycles. For an integer  $k \geq 3$ , let

$$\beta(k) = \sup_w \beta(w, C_k),$$

where  $w$  is allowed to vary across all  $n$  and all weight functions satisfying Conditions 1 and 2.

**Lemma 1** (Cox and Martin [3]). *For all  $k \geq 3$ , the number of  $2k$ -cycles in a planar graph is at most*

$$\beta(k)n^k + o(n^k).$$

Cox and Martin conjectured that  $\beta(k) \leq \frac{1}{k^k}$ . By Lemma 1 such a bound immediately implies Theorem 1. In Section 3, we prove that this bound indeed holds.

**Theorem 2.** *For all  $k \geq 3$ ,*

$$\beta(k) \leq \frac{1}{k^k}.$$

*Equality is attained only for weight functions satisfying  $w(e) = \frac{1}{k}$  for  $e \in E(C)$  and  $w(e) = 0$  otherwise, where  $C$  is a fixed cycle of length  $k$  of  $K_n$ .*

### 3 Proof of Theorem 2

*Proof.* Let us fix an integer  $n$ , a complete graph  $K_n$  and a function  $w$  satisfying Conditions 1 and 2. Let us assume  $w$  maximizes  $\sum_{C_k \subseteq K_n} p_w(C_k)$ . Let  $P_j$  be a path with  $j$  vertices. A  $(j+2)$ -vertex path with terminal vertices  $u$  and  $v$  is denoted by  $uP_jv$ . For vertices  $u$  and  $v$ , a subgraph  $H$  of  $K_n$  and an integer  $j$  such that  $2 \leq j \leq n$ , we define

$$f_H(j, u, v) = \sum_{uP_{j-2}v \subseteq H} p_w(uP_{j-2}v),$$

and

$$f_H(j, u) = \sum_{v \in V(H) \setminus \{u\}} f(j, u, v).$$

In the case when  $H$  is the complete graph  $K_n$  we simply write  $f(j, u, v)$  and  $f(j, u)$ . The following lemma will be essential in the proof of Theorem 2. Related lemmas were also deduced in the original paper of Cox and Martin [3] (see Lemmas 4.5 and 4.6 in their paper), and both approaches can be used to deduce an upper bound of  $1/k$  on the weight of every edge.

**Lemma 2.** *Let  $k \geq 2$ , and let  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$  be distinct edges of  $K_n$  such that  $w(e_1) > 0$  and  $w(e_2) > 0$ . Then we have  $f(k, u_1, v_1) = f(k, u_2, v_2)$ .*

*Proof of Lemma 2.* Omitted for space. Full proofs can be found in the Arxiv preprint of the same title.  $\square$

From Lemma 2, for an edge  $uv$  with non-zero weight  $w(uv) > 0$  we may assume  $f(j, u, v) = \mu$  for some fixed constant  $\mu$ . Hence we have

$$\sum_{C_k \subseteq K_n} p_w(C_k) = \frac{1}{k} \sum_{uv \in E(K_n)} w(uv) f(j, u, v) = \frac{\mu}{k} \sum_{uv \in E(K_n)} w(uv) = \frac{\mu}{k}. \quad (1)$$

Furthermore  $w(e) \leq 1/k$  for every edge  $e \in E(K_n)$ . Indeed,

$$w(e)\mu = \sum_{e \in C_k} p_w(C_k) \leq \sum_{C_k \subseteq K_n} p_w(C_k) = \frac{\mu}{k}.$$

For any subgraph  $G$  of  $K_n$  and any vertex  $v \in V(K_n)$  we denote  $\sum_{u \in V(G)} w(uv)$  by  $d_G(v)$ . Furthermore, for a vertex set  $S \subseteq V(G)$ , we denote the graph  $G[V(G) \setminus S]$  by  $G \setminus S$ . Also for an edge  $e \in E(G)$ , the graph with vertex set  $V(G)$  and edge set  $E(G) \setminus \{e\}$  is denoted by  $G \setminus e$ .

**Lemma 3.** *For a fixed integer  $r$  such that  $3 \leq r \leq n$  and distinct vertices  $v_1$  and  $u$  there exists a sequence  $v_2, v_3, \dots, v_{r-1}$  of distinct vertices such that for every integer  $t$  satisfying  $1 \leq t \leq r-1$ , where  $G_1 = K_n \setminus v_1u$  and  $G_i = K_n \setminus \{v_1, v_2, \dots, v_{i-1}\}$  for every  $i = 2, 3, \dots, r-1$ , we have*

$$f_{G_1}(r, v_1, u) \leq d_{G_1}(v_1) d_{G_2}(v_2) \cdots d_{G_{t-1}}(v_{t-1}) f_{G_t}(r-t+1, v_t, u).$$

*Proof.* Omitted for space.  $\square$

**Lemma 4.** *For any non-negative weight function  $w : E(K_n) \rightarrow \mathbb{R}$  and for every vertex  $v$  and integer  $r$  with  $2 \leq r \leq n$ , we have*

$$f(r, v) \leq \left( \frac{\sum_{e \in E(K_n)} w(e)}{r-1} \right)^{r-1}.$$

**Remark 1.** In Lemma 4, we do not require that  $\sum_{e \in E(K_n)} w(e) = 1$ , only that the weights are non-negative.

*Proof.* Omitted for space.  $\square$

In order to finish the proof of Theorem 2 it is sufficient to show that  $\mu \leq \frac{1}{k^{k-1}}$  by (1).

Choose an edge  $v_0v_1$  with the maximum weight  $w(v_0v_1)$ . Let us denote the graph  $K_n \setminus v_0v_1$  by  $G_1$ . By Lemma 3 we have a sequence of vertices  $v_2, v_3, \dots, v_{k-1} \in V(K_n)$  satisfying the following inequality for every  $t$  where  $1 \leq t \leq k-1$  and  $G_i = K_n \setminus \{v_1, v_2, \dots, v_{i-1}\}$  for all  $i \in \{2, 3, \dots, r-1\}$ :

$$f_{G_1}(k, v_1, v_0) \leq d_{G_1}(v_1)d_{G_2}(v_2) \cdots d_{G_{t-1}}(v_{t-1})f_{G_t}(k-t+1, v_t, v_0). \quad (2)$$

Here we distinguish the following two cases.

**Case 1:** Suppose that  $d_{G_1}(v_1) + d_{G_2}(v_2) + \cdots + d_{G_{k-2}}(v_{k-2}) \leq \frac{k-2}{k}$ . Then by the inequality of the arithmetic and geometric means we have

$$\prod_{i=1}^{k-2} d_{G_i}(v_i) \leq \left( \frac{\sum_{i=1}^{k-2} d_{G_i}(v_i)}{k-2} \right)^{k-2} \leq \frac{1}{k^{k-2}}.$$

From (2) we obtain the desired inequality

$$\mu = f_{G_1}(k, v_1, v_0) \leq \left( \prod_{i=1}^{k-2} d_{G_i}(v_i) \right) \cdot f_{G_{k-1}}(2, v_{k-1}, v_0) \leq \frac{1}{k^{k-2}} \frac{1}{k} \leq \frac{1}{k^{k-1}}.$$

Even more the inequality holds with equality if and only if  $w(v_0v_1) = w(v_1v_2) = \cdots = w(v_{k-2}v_{k-1}) = w(v_{k-1}v_0) = 1/k$  (here we use that for all  $e$ ,  $w(e) \leq 1/k$ ). Therefore equality is attained in Theorem 2 only for weight functions satisfying  $w(e) = \frac{1}{k}$  for  $e \in E(C)$  and  $w(e) = 0$  otherwise, where  $C$  is a fixed cycle of length  $k$  of  $K_n$ .

**Case 2:** Suppose that  $d_{G_1}(v_1) + d_{G_2}(v_2) + \cdots + d_{G_{k-2}}(v_{k-2}) > \frac{k-2}{k}$ . Proof of this case is omitted for space.  $\square$

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