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# A general approach to transversal versions of Dirac-type theorems

(Extended abstract)

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**Abstract**

Given a collection of hypergraphs $\mathbf{H} = (H_1, \dots, H_m)$ with the same vertex set, an $m$-edge graph $F \subset \cup_{i\in[m]} H_i$ is a transversal if there is a bijection $\phi : E(F) \to [m]$ such that $e \in E(H_{\phi(e)})$ for each $e \in E(F)$. How large does the minimum degree of each $H_i$ need to be so that $\mathbf{H}$ necessarily contains a copy of $F$ that is a transversal? Each $H_i$ in the collection could be the same hypergraph, hence the minimum degree of each $H_i$ needs to be large enough to ensure that $F \subseteq H_i$. Since its general introduction by Joos and Kim [Bull. Lond. Math. Soc., 2020, 52(3): 498–504], a growing body of work has shown that in many cases this lower bound is tight. In this paper, we give a unified approach to this problem by providing a widely applicable sufficient condition for this lower bound to be asymptotically tight. This is general enough to recover many previous results in the area and obtain novel transversal variants of several classical Dirac-type results for (powers of) Hamilton cycles. For example, we derive that any collection of $rn$ graphs on an $n$-vertex set, each with minimum degree at least $(r/(r+1) + o(1))n$, contains a transversal copy of the $r$-th power of a Hamilton

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cycle. This can be viewed as a rainbow version of the Pósa-Seymour conjecture.

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# 1 Introduction

Given an integer $m \geq 1$, we say that $\mathbf{H} = (H_1, \dots, H_m)$ is a *hypergraph collection on vertex set* $V$ if, for each $i \in [m]$, the hypergraph $H_i$ has vertex set $V$. We call the collection a *graph collection* if each hypergraph in the collection has uniformity two. Given an $m$-edge hypergraph $F$ on $V$, we say that $\mathbf{H}$ has a *transversal* copy of $F$ if there is a bijection $\phi\colon E(F) \to [m]$ such that $e \in H_{\phi(e)}$ for each $e \in E(F)$. We will also use the adjective *rainbow* for a transversal copy of $F$. Indeed, we can think of the edges of hypergraph $H_i$ to be coloured with colour $i$ and, in this framework, a transversal copy of $F$ is a copy of $F$ in $\bigcup_{i\in[m]} H_i$ with edges of pairwise distinct colours. We are interested in the following general question formulated originally by Joos and Kim [6].

**Question 1.** Let $F$ be an $m$-edge hypergraph with vertex set $V$, $\mathcal{H}$ be a family of hypergraphs, and $\mathbf{H} = (H_1, \dots, H_m)$ be a hypergraph collection on vertex set $V$ with $H_i \in \mathcal{H}$ for each $i \in [m]$. Which conditions on $\mathcal{H}$ guarantee a transversal copy of $F$ in $\mathbf{H}$?

By taking $H_1 = H_2 = \cdots = H_m$, it is clear that such a property needs to guarantee that each hypergraph in $\mathcal{H}$ contains $F$ as a subhypergraph. However, this alone is not always sufficient, not even asymptotically. For example, Aharoni, DeVos, de la Maza, Montejano and Šámal [1] showed that if $\mathbf{G} = (G_1, G_2, G_3)$ is a graph collection on $[n]$ with $e(G_i) > (\frac{26-2\sqrt{7}}{81})n^2$ for each $i \in [3]$, then $\mathbf{G}$ contains a transversal which is a triangle. As shown in [1], the constant $\frac{26-2\sqrt{7}}{81} > 1/4$ is optimal. On the other hand, Mantel's theorem states that any graph with at least $n^2/4$ edges must contain a triangle.

Instead of a lower bound on the total number of edges, it is also natural to investigate what can be guaranteed with a lower bound on the minimum degree. It turns out that even in this more restrictive setting, there can be a discrepancy between the uncoloured and the rainbow versions of the problem. To make this more precise, we give the following two definitions, where, for a $k$-uniform hypergraph $H$ and $1 \leq d < k$, we let $\delta_d(H)$ be the minimum number of edges of $H$ that any set of $d$ vertices of $V(H)$ is contained in. Moreover, for a hypergraph collection $\mathbf{H} = (H_1, \dots, H_m)$, we denote $|\mathbf{H}| = m$ and $\delta_d(\mathbf{H}) = \min_{i\in[m]} \delta_d(H_i)$.

**Definition 1.1** (Uncoloured minimum degree threshold)**.** Let $\mathcal{F}$ be an infinite family of $k$-uniform hypergraphs. By $\delta_{\mathcal{F},d}$ we denote, if it exists, the smallest real number $\delta$ such that for all $\alpha > 0$ and for all but finitely many $F \in \mathcal{F}$ the following holds. Let $n = |V(F)|$ and $H$ be any $n$-vertex $k$-uniform hypergraph with $\delta_d(H) \geq (\delta + \alpha)n^{k-d}$. Then $H$ contains a copy of $F$.

For example, if $\mathcal{F}$ is the family of graphs consisting of a cycle on $n$ vertices for each $n \in \mathbb{N}$, then we have $\delta_{\mathcal{F},1} = 1/2$. Indeed, this follows from Dirac's theorem which states that any graph with minimum degree at least $n/2$ has a Hamilton cycle.

**Definition 1.2** (Rainbow minimum degree threshold)**.** Let $\mathcal{F}$ be an infinite family of $k$-uniform hypergraphs. By $\delta^{\mathrm{rb}}_{\mathcal{F},d}$ we denote, if it exists, the smallest real number $\delta$ such that for all $\alpha > 0$ and for all but finitely many $F \in \mathcal{F}$ the following holds. Let $n = |V(F)|$ and $\mathbf{H}$ be any $k$-uniform hypergraph collection on $n$ vertices with $|\mathbf{H}| = |E(F)|$ and $\delta_d(\mathbf{H}) \geq (\delta + \alpha)n^{k-d}$. Then $\mathbf{H}$ contains a transversal copy of $F$.

If the two values are well-defined, it must be that $\delta^{\mathrm{rb}}_{\mathcal{F},d} \geq \delta_{\mathcal{F},d}$. Indeed, if $H$ contains no copy of $F$, the collection $\mathbf{H}$ consisting of $|E(F)|$ copies of $H$ does not contain a transversal copy of $H$ either. However, Montgomery, Müyesser, and Pehova [11] made the following observation which shows that $\delta^{\mathrm{rb}}_{\mathcal{F},d}$ can be much larger than $\delta_{\mathcal{F},d}$. Set $\mathcal{F} = \{k \times (K_{2,3} \cup C_4) \colon k \in \mathbb{N}\}$ where $k \times G$ denotes the graph obtained by taking $k$ vertex-disjoint copies of $G$. It follows from a result of Kühn and Osthus [7] that $\delta_{\mathcal{F},1} = 4/9$. Consider the graph collection $\mathbf{H} = (H_1, \ldots, H_m)$ on $V$ obtained in the following way. Partition $V$ into two almost equal vertex subsets, say $A$ and $B$, and suppose that $H_1 = H_2 = \cdots = H_{m-1}$ are all disjoint unions of a clique on $A$ and a clique on $B$. Suppose that $H_m$ is a complete bipartite graph between $A$ and $B$. Observe that each $H_i$ in this resulting graph collection has minimum degree $\lfloor |V|/2 \rfloor$. Further observe that if $\mathbf{H}$ contains a transversal copy of some $F \in \mathcal{F}$, the edge of $K_{2,3}$ or $C_4$ that gets copied to an edge of $H_m$ would be a bridge (an edge whose removal disconnects the graph) of $F$. However, neither $K_{2,3}$ nor $C_4$ contains a bridge. Hence, $\delta^{\mathrm{rb}}_{\mathcal{F},d} \geq 1/2$.

On the other hand, there are many natural instances where $\delta^{\mathrm{rb}}_{\mathcal{F},d} = \delta_{\mathcal{F},d}$. When this equality holds, we say that the corresponding family $\mathcal{F}$ is *d-colour-blind*, or just *colour-blind* in the case $\mathcal{F}$ is a family of graphs (and $d = 1$). For example, Joos and Kim [6], improving a result of Cheng, Wang, and Zhao [4] and confirming a conjecture of Aharoni [1], showed that, if $n \geq 3$, then any $n$-vertex graph collection $\mathbf{G} = (G_1, \ldots, G_n)$ with $\delta(G_i) \geq n/2$ for each $i \in [n]$ has a transversal copy of a Hamilton cycle. This generalises Dirac's classical theorem and implies that the family $\mathcal{F}$ of $n$-cycles is colour-blind[1]. There are many more families of colour-blind (hyper)graphs. In particular, matchings [2, 8, 9, 10], Hamilton $\ell$-cycles [3], factors [2, 11], and spanning trees [11] have been extensively studied. We recall that for $1 \leq \ell < k$, a $k$-uniform hypergraph is called an $\ell$-cycle if its vertices can be ordered cyclically such that each of its edges consists of $k$ consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly $\ell$ vertices.

Building on techniques introduced by Montgomery, Müyesser, and Pehova [11], we give a widely applicable sufficient condition for a family of hypergraphs $\mathcal{F}$ to be colour-blind. Our condition gives a unified proof of several of the aforementioned results, as well as many new rainbow Dirac-type results. The following theorem lists some applications, though we believe that our setting can capture even more families of hypergraphs.

[1]In fact, in this particular case, the corresponding thresholds are *exactly* the same, and there is no need for an error term.

**Theorem 1.3.** *The following families of hypergraphs are all $d$-colour-blind.*

*(A) The family of the $r$-th powers of Hamilton cycles for fixed $r \geq 2$ (and $d = 1$).*

*(B) The family of $k$-uniform Hamilton $\ell$-cycles for the following ranges of $k$, $\ell$, and $d$.*

*(B1) $1 < \ell < k/2$ and $d = k - 2$;*

*(B2) $1 \leq \ell < k/2$ or $\ell = k - 1$, and $d = k - 1$;*

*(B3) $\ell = k/2$ and $k/2 < d \leq k - 1$ with $k$ even.*

**Remark 1.** Theorem 1.3 (B2) when $\ell = k-1$ was originally proven by Cheng, Han, Wang, Wang, and Yang [3], who raised the problem of obtaining the rainbow minimum degree threshold for a wider range of $\ell \in [k-2]$. Moreover, the case of Hamilton cycles in graphs (i.e. $k = 2$ and $d = \ell = 1$) was previously proven by Cheng, Wang, and Zhao [4] (and their result was sharpened by Joos and Kim [6]).

# 2 Towards the statement of the main theorem

The precise statement of our main theorem is quite technical, therefore we provide some intuition here and refer the interested reader to the arXiv preprint [5]. Firstly, we look at hypergraph families $\mathcal{F}$ with a 'cyclic' structure. That is, we assume there exists a $k$-uniform hypergraph $\mathcal{A}$ such that all $F \in \mathcal{F}$ can be obtained by gluing several copies of $\mathcal{A}$ in a Hamilton cycle fashion, and in this case we say that $F$ is a Hamilton $\mathcal{A}$-*cycle*. Similarly, an $\mathcal{A}$-*chain* is a graph obtained by gluing several copies of $\mathcal{A}$ in a path-like fashion. Moreover, the first (resp. last) copy of $\mathcal{A}$ in that chain is called the *start* (resp. the *end*) of the chain. For example, for $k$-uniform Hamilton cycles, $\mathcal{A}$ would be a single $k$-uniform edge (see Figure 1), whereas for the $r$-th power of a Hamilton cycle, $\mathcal{A}$ would be a a clique on $r$ vertices (see Figure 2).

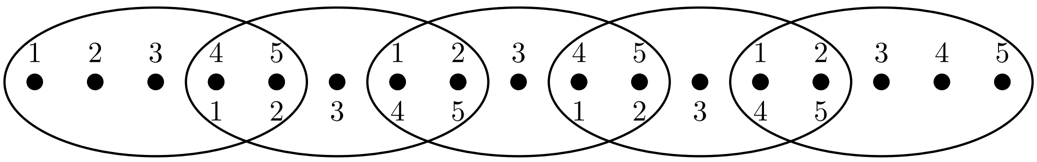


Figure 1: A 5-uniform 2-path is an $\mathcal{A}$-chain, with $\mathcal{A}$ being (any ordering of) a single 5-uniform edge. The numbering of the vertices in each edge denotes the (ordered) isomorphism between that edge and $\mathcal{A}$.

In the uncoloured setting, most of the well-studied problems fit into this framework, including everything listed in Theorem 1.3. A common framework for embedding such hypergraphs with cyclic structure is the absorption method. Our main result essentially states that if there is an absorption-based proof that $\delta$ is the uncoloured minimum $d$-degree threshold for some $\mathcal{F}$ with cyclic structure, then the rainbow minimum $d$-degree threshold of $\mathcal{F}$ is equal to $\delta$. While some partial progress towards such an abstract statement was

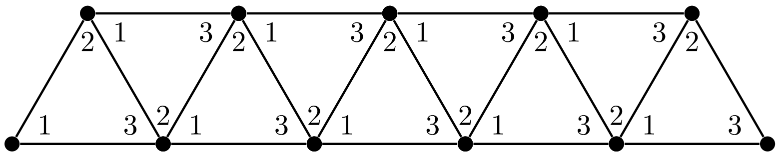


Figure 2: The square of a path is an $\mathcal{A}$-chain, with $\mathcal{A}$ being (any ordering of) a triangle.

already made in [11], our approach does not require the need to make ad-hoc strengthenings to the uncoloured version of the result, allowing for a very short proof of Theorem 1.3. To achieve this, we codify what it means for there to be streamlined absorption proof for the uncoloured result, and we use the existence of such a proof as a black-box. We do so through two properties: **Ab** and **Con**. Property **Ab** states that every $k$-uniform hypergraph with minimum $d$-degree at least $(\delta + \alpha)n^{k-d}$ contains an absorber for $\mathcal{A}$, i.e. a set of vertices which can absorb any small set of vertices into an $\mathcal{A}$-chain. Property **Con** states that in every $k$-uniform hypergraph with minimum $d$-degree at least $(\delta + \alpha)n^{k-d}$, any two copies $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{A}$ can be connected into an $\mathcal{A}$-chain of bounded length with start $\mathcal{S}$ and end $\mathcal{T}$.

In addition to properties **Ab** and **Con** which guarantee we can rely on a streamlined absorption proof for the uncoloured result, our main theorem assumes another property, which we call property **Fac**. One reason why transversal versions of Dirac-type results are more difficult is that every single hypergraph in the collection as well as every single vertex of the host graph needs to be utilised in the target spanning structure (the transversal). This is crucial as demonstrated by the construction given after Definition 1.2. In this construction, the possibility of finding a transversal copy of $\mathcal{F}$ is ruled out by showing that a particular graph in the collection (namely the hypergraph $H_m$) cannot be used in a transversal copy of a $K_{2,3}$ or $C_4$. Therefore, in addition to some properties which are related to the uncoloured case and where colours do not play any role, we require a property concerning the coloured case which we call **Fac**. This roughly states that vertex-disjoint copies of $\mathcal{A}$ (the building block of the hypergraph we are trying to find) can be found in a rainbow fashion using a fixed, adversarially specified set of hypergraphs from the collection. This ensures that we never get stuck while trying to use up every single colour/hypergraph that we start with.

Our main theorem claims that if properties **Ab**, **Con** and **Fac** hold for $\mathcal{A}$, then the family of Hamilton $\mathcal{A}$-cycles is $d$-colour-blind. In all our applications (see Theorem 1.3), in order to ensure that properties **Ab** and **Con** hold, we rely on existing lemmas in the literature without having to do any extra work. Moreover, when $\mathcal{A}$ is a single edge (as it is the case for Theorem 1.3 (B)), the property **Fac** is trivial to check. For powers of Hamilton cycles, however, this property is more delicate and, in order to verify it, we rely on a non-trivial coloured property from [11].

# 3 Proof overview

We will now attempt to give a self-contained account of the main ideas of our proof strategy. For the purposes of the proof sketch, it will be conceptually (and notationally) simpler to imagine that we are trying to prove that the family of (2-uniform) Hamilton cycles is colour-blind. Observe that a Hamilton cycle is an $\mathcal{A}$-cycle with $\mathcal{A}$ being an edge.

**Proposition 3.1** (Theorem 2 in [4]). *For any $\alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds. Let $\mathbf{G}$ be a graph collection on vertex set $[n]$ with $|\mathbf{G}| = n$ and $\delta(\mathbf{G}) \geq (1/2 + \alpha)n$. Then $\mathbf{G}$ contains a transversal copy of a Hamilton cycle.*

The basic premise of our approach, which is shared with [11], is that Proposition 3.1 becomes significantly easier to prove if we assume that $|\mathbf{G}| = (1 + o(1))n$, that is, if we have a bit more colours than we need to find a rainbow Hamilton cycle on $n$ vertices. Thus, relying on the hypergraph analogue of Lemma 3.4 from [11], it is enough to show the following.

**Proposition 3.2.** *Let $1/n \ll \zeta \ll \kappa, \alpha$. Let $\mathbf{G}$ be a graph collection on $[n]$ with $|\mathbf{G}| = (1 + \kappa - \zeta)n$ and $\delta(\mathbf{G}) \geq (1/2 + \alpha)n$. Let $a, b \in [n]$ be distinct vertices. Then, $\mathbf{G}$ contains a rainbow Hamilton path with $a$ and $b$ as its endpoints, using every colour $G_i$ with $i \in [(1 - \zeta)n]$.*

Unfortunately, due to the technicalities present in the statement, Proposition 3.2 is far from trivial to show. Most of the novelty in the proof of our main theorem is the way we approach Proposition 3.2 for arbitrary $\mathcal{A}$-chains satisfying **Ab**, **Con**, and **Fac**. We now proceed to explain briefly how we achieve this, and how the three properties come in handy.

Firstly, in the setting of Proposition 3.2, it is quite easy to find a few rainbow paths using most of the colours from the set $[(1 - \zeta)n]$. Below is a formal statement of a version of this for arbitrary $\mathcal{A}$-chains, where we write $s(\mathcal{A}) \cdot n$ for the number of edges of an $\mathcal{A}$-cycle spanning $n$ vertices.

**Lemma 3.3.** *Let $1/n \ll 1/T \ll \omega, \alpha$. Let $\mathcal{A}$ be $k$-uniform graph and $d \in [k - 1]$. Let $\delta$ be the minimum $d$-degree threshold for the containment of a Hamilton $\mathcal{A}$-cycle. Let $\mathbf{H}$ be a $k$-uniform hypergraph collection on $[n]$ with $\delta_d(\mathbf{H}) \geq (\delta + \alpha)n^{k-d}$ and suppose that $|\mathbf{H}| \geq s(\mathcal{A}) \cdot n$. Then $\mathbf{H}$ contains a rainbow collection of $T$-many pairwise vertex-disjoint $\mathcal{A}$-chains covering all but at most $\omega n$ vertices of $\mathbf{H}$.*

Although it is easy to use most of the colours coming from a colour set using the above result, a challenge in Proposition 3.2 is that we need to use all of the colours coming from the set $[(1 - \zeta)n]$. As we are currently concerned with the case when $\mathcal{A}$ consists of a single edge, this will not be a major issue. Indeed, using the minimum degree condition on each of the colours, we can greedily find rainbow matchings using small colour subsets of $[(1 - \zeta)n]$. For arbitrary $\mathcal{A}$, we would like to proceed in the same way; however, say when $\mathcal{A}$ is a triangle, the situation becomes considerably more complicated. This is why the property **Fac** is built into the assumptions of the main theorem.

Our ultimate goal is to build a single $\mathcal{A}$-chain connecting specific ends, not just a collection of $\mathcal{A}$-chains. Hence, we rely on the property **Con** to connect the ends of the paths we obtained via Lemma 3.3 (as well as the greedy matching we found for the purpose of exhausting a specific colour set). An issue is that **Con** is an uncoloured property, whereas we would like to connect these ends in a rainbow manner. Here we rely on the following trick: in hypergraph collections where each hypergraph has good minimum $d$-degree conditions, we can pass down to the following auxiliary hypergraph $\mathcal{K}$ which also has good minimum $d$-degree conditions. An edge appears in $\mathcal{K}$ if and only if that edge belongs to $\Omega(n)$ many colours in the original hypergraph collection. We can use the property **Con** on $\mathcal{K}$ to connect ends via short uncoloured paths, and later assign greedily one of the many available colours to the edges on this path.

As is the case with many absorption-based arguments, the short connecting paths we find will be contained in a pre-selected random set. After all the connections are made, there will remain many unused vertices inside this random set. To include these vertices inside a path, we use the property **Ab**. Similarly to **Con**, property **Ab** is an uncoloured property, but we can use again the trick of passing down to an appropriately chosen auxiliary hypergraph.

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