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Refined list version of Hadwiger's Conjecture

(Extended abstract)

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Abstract

Assume $\lambda = \{k_1, k_2, \dots, k_q\}$ is a partition of $k_\lambda = \sum_{i=1}^{q} k_i$. A λ-list assignment of G is a k_λ-list assignment L of G such that the colour set $\bigcup_{v \in V(G)} L(v)$ can be partitioned into $|\lambda| = q$ sets $C_1, C_2, \dots, C_q$ such that for each i and each vertex v of G, $|L(v) \cap C_i| \geq k_i$. We say G is *λ-choosable* if G is L-colourable for any λ-list assignment L of G. The concept of λ-choosability is a refinement of choosability that puts k-choosability and k-colourability in the same framework. If $|\lambda|$ is close to k_λ, then λ-choosability is close to k_λ-colourability; if $|\lambda|$ is close to 1, then λ-choosability is close to k_λ-choosability. This paper studies Hadwiger's Conjecture in the context of λ-choosability. Hadwiger's Conjecture is equivalent to saying that every K_t-minor-free graph is $\{1 \star (t-1)\}$-choosable for any positive integer t. We prove that for $t \geq 5$, for any partition λ of $t-1$ other than $\{1 \star (t-1)\}$, there is a K_t-minor-free graph G that is not λ-choosable. We then construct several types of K_t-minor-free graphs that are not λ-choosable, where $k_\lambda - (t-1)$ gets larger as $k_\lambda - |\lambda|$ gets larger.

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1 Introduction

Given graphs H and G, we say H is *a minor* of G (or G has *an H-minor*) if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. Let K_t

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be the t-vertex complete graph. A graph G is *K_t-minor-free* if G has no K_t-minor. In 1943, Hadwiger [8] conjectured the following upper bound on the chromatic number of K_t-minor-free graphs:

Conjecture 1 (Hadwiger's Conjecture)**.** *For every integer $t \geq 1$, every K_t-minor-free graph is $(t-1)$-colourable.*

This conjecture is a deep generalization of the Four Colour Theorem, and has motivated many developments in graph colouring and graph minor theory. Hadwiger [8] and Dirac [6] independently showed that Hadwiger's Conjecture holds for $t \leq 4$. Wagner [27] proved that for $t = 5$ the conjecture is equivalent to the Four Colour Theorem, which was subsequently proved by Appel, Haken and Koch [2,3] and Robertson, Sanders, Seymour and Thomas [20], both using extensive computer assistance. Robertson, Seymour and Thomas [21] went one step further and proved Hadwiger's Conjecture for $t = 6$, also by reducing it to the Four Colour Theorem. The conjecture for $t \geq 7$ is open and seems to be extremely challenging. For more on Hadwiger's Conjecture, see the survey of Seymour [23].

The evident difficulty of Hadwiger's Conjecture has inspired many researchers to study the following natural weakening (cf. [9, 10, 19]):

Conjecture 2 (Linear Hadwiger's Conjecture)**.** *There exists a constant $C > 0$ such that for every integer $t \geq 1$, every K_t-minor-free graph is Ct-colourable.*

For many decades, the best general upper bound on the chromatic number of K_t-minor-free graphs was $O(t\sqrt{\log t})$, which was proved independently by Kostochka [12, 13] and Thomason [24] in the 1980s. In 2019, Norine, Postle and Song [15] broke this barrier, and proved that the maximum chromatic number of K_t-minor-free graphs is in $O(t(\log t)^{1/4+o(1)})$. Following a series of improvements, [14, 16–18] the best known bound is $O(t \log\log t)$ due to Delcourt and Postle [5].

A list assignment of a graph G is a mapping L that assigns to each vertex v of G a set $L(v)$ of permissible colours. An *L-colouring* of G is a proper colouring f of G such that for each vertex v of G, $f(v) \in L(v)$. We say G is *L-colourable* if G has an L-colouring. A *k-list assignment* of G is a list assignment L with $|L(v)| \geq k$ for each vertex v. We say G is *k-choosable* if G is L-colourable for any k-list assignment L of G. The *choice-number* of G is the minimum integer k such that G is k-choosable.

Hadwiger's Conjecture is also widely considered in the setting of list colourings. Voigt [26] constructed planar graphs (hence K_5-minor-free) with choice-number 5. Hence the list version of Hadwiger's Conjecture is false. Nevertheless, the list version of Linear Hadwiger's Conjecture, proposed by Kawarabayashi and Mohar [10] in 2007, remains open.

Conjecture 3 (List Hadwiger's Conjecture)**.** *There exists a constant $C > 0$ such that for every integer $t \geq 1$, every K_t-minor-free graph is Ct-choosable.*

The current state-of-the-art upper bound on the choice-number of K_t-minor-free graphs is $O(t(\log\log t)^6)$ [18]. If Conjecture 3 is true, then a natural problem is to determine the minimum value of C. Barát, Joret and Wood [4] constructed K_t-minor-free graphs that are not $4(t-3)/3$-choosable, implying $C \geq \frac{4}{3}$ in Conjecture 3. Improving upon this result, Steiner [22] recently proved that the maximum choice-number of K_t-minor-free graphs is at least $2t - o(t)$, and hence $C \geq 2$ in Conjecture 3.

This paper considers Hadwiger's Conjecture in the context of λ-choosability, which was introduced by Zhu [28]. In general, k-colourability and k-choosability behave very differently. Indeed, bipartite graphs can have arbitrary large choice-number. λ-choosability is a refinement of the concept of choosability that puts k-choosability and k-colourability in the same framework and considers a more complex hierarchy of colouring parameters.

Definition 1. *Let $\lambda = \{k_1, k_2, \dots, k_q\}$ be a multiset of positive integers. Let $k_\lambda = \sum_{i=1}^{q} k_i$ and $|\lambda| = q$. A* λ-list assignment *of G is a list assignment L such that the colour set $\bigcup_{v \in V(G)} L(v)$ can be partitioned into q sets $C_1, C_2, \dots, C_q$ such that for each i and each vertex v of G, $|L(v) \cap C_i| \geq k_i$. We say G is* λ-choosable *if G is L-colourable for any λ-list assignment L of G.*

Note that for each vertex v, $|L(v)| \geq \sum_{i=1}^{q} k_i = k_\lambda$. So a λ-list assignment L is a k_λ-list assignment with some restrictions on the set of possible lists.

For a positive integer a, let $m_\lambda(a)$ be the multiplicity o f a in λ. If $m_\lambda(a) = m$, then instead of writing m times the integer a, we write $a \star m$. For example, $\lambda = \{1 \star k_1, 2 \star k_2, 3\}$ means that λ is the multiset consisting of k_1 copies of 1, k_2 copies of 2 and one copy of 3. If $\lambda = \{k\}$, then λ-choosability is the same as k-choosability; if $\lambda = \{1 \star k\}$, then λ-choosability is equivalent to k-colourability. So the concept of λ-choosability puts k-choosability and k-colourability in the same framework.

For $\lambda = \{k_1, k_2, \dots, k_q\}$ and $\lambda' = \{k'_1, k'_2, \dots, k'_p\}$, we say λ' is a *refinement* of λ if $p \geq q$ and there is a partition $I_1, I_2, \dots, I_q$ of $\{1, 2, \dots, p\}$ such that $\sum_{j \in I_t} k'_j = k_t$ for $t = 1, 2, \dots, q$. We say λ' is obtained from λ by *increasing* some parts if $p = q$ and $k_t \leq k'_t$ for $t = 1, 2, \dots, q$. We write $\lambda \leq \lambda'$ if λ' is obtained from a refinement of λ by increasing some parts. It follows from the definitions that if $\lambda \leq \lambda'$, then every λ-choosable graph is λ'-choosable. Conversely, Zhu [28] proved that if $\lambda \nleq \lambda'$, then there is a λ-choosable graph that is not λ'-choosable. In particular, λ-choosability implies k_λ-colourability, and if $\lambda \neq \{1 \star k_\lambda\}$, then there are k_λ-colourable graphs that are not λ-choosable.

All the partitions λ of a positive integer k are sandwiched between $\{k\}$ and $\{1 \star k\}$ in the above order. As observed above, $\{k\}$-choosability is the same as k-choosability, and $\{1 \star k\}$-choosability is equivalent to k-colourability. By considering each partition λ of k, λ-choosability provides a complex hierarchy of colouring parameters that interpolate between k-colourability and k-choosability.

The framework of λ-choosability provides room to explore strengthenings of colourability and choosability results. For example, Kermnitz and Voigt [11] proved that there are planar graphs that are not $\{1, 1, 2\}$-choosable. This result strengthens Voigt's result that there are non-4-choosable planar graphs, and shows that the Four Colour Theorem is sharp in the sense that for any partition λ of 4 other than $\{1 \star 4\}$, there is a planar graph that is not λ-choosable. This paper considers Hadwiger's Conjecture in the context of λ-choosability.

2 Results

This paper constructs several examples of K_t-minor-free graphs that are not λ-choosable where $k_\lambda \geq t-1$ and q is close to k_λ. In particular, if the multiplcity of 1 in λ is large enough, then the number of parts of λ will be close to k_λ.

First we strengthen the above-mentioned result of Kermnitz and Voigt to K_t-minor-free graphs for $t \geq 5$ as follows:

Theorem 1. *For every integer $t \geq 5$, there exists a K_t-minor-free graph that is not $\{1 \star (t-3), 2\}$-choosable.*

If λ is a partition of $t-1$ other than $\{1 \star (t-1)\}$, then $\{1 \star (t-3), 2\}$ is a refinement of λ. Hence we have the following corollary.

Corollary 2. *If λ is a partition of $t-1$ other than $\{1 \star (t-1)\}$, then there is a K_t-minor-free graph that is not λ-choosable.*

For a multiset λ of positive integers, let $h(\lambda)$ be the maximum t such that every K_t-minor-free graph is λ-choosable. Since $K_{k_\lambda+1}$ is not k_λ-colourable and hence not λ-choosable, we know that $h(\lambda) \leq k_\lambda + 1$.

For a multiset λ of positive integers, $k_\lambda - |\lambda|$ measures the "distance" of λ-choosability from k_λ-colourability. Hadwiger's Conjecture says that if $k_\lambda - |\lambda| = 0$, then $h(\lambda) = k_\lambda + 1$. By Theorem 1, if $k_\lambda - |\lambda| \geq 1$, then $h(\lambda) \leq k_\lambda$, provided that $k_\lambda \geq 5$. It seems natural that if $k_\lambda - |\lambda|$ gets bigger, then $k_\lambda - h(\lambda)$ also gets bigger, provided that k_λ is sufficiently large. The next result shows this is true for various λ.

Theorem 3. *For each integer $a \geq 0$, there exists an integer $t_1 = t_1(a)$ such that for every integer $t \geq t_1$, there exists a K_t-minor-free graph that is not $\{1 \star (t-2a-6), 3a+6\}$-choosable.*

For the λ in Theorem 3, $k_\lambda = t+a$, $h(\lambda) \leq t-1$ and $|\lambda| = t-(2a+5)$. As $k_\lambda - |\lambda| = 3a+5$ tends to infinity, the difference $k_\lambda - h(\lambda) \geq a+1$ also tends to infinity, provided that $k_\lambda \geq \phi(k_\lambda - |\lambda|)$, where ϕ is a certain given function. It remains open whether such a conclusion holds for all λ. We conjecture a positive answer.

Conjecture 4. *There are functions $\phi, \psi : \mathbb{N} \to \mathbb{N}$ for which the following hold:*

- $\lim_{n\to\infty} \psi(n) = \infty$.
- *For any multiset λ of positive integers, if $k_\lambda \geq \phi(k_\lambda - |\lambda|)$, then $k_\lambda - h(\lambda) \geq \psi(k_\lambda - |\lambda|)$.*

It is easy to see that if $k_\lambda - |\lambda| = b$, then $\{1 \star (k_\lambda - 2b'), 2 \star b'\}$ is a refinement of λ, where $b \geq b' \geq b/2$. Thus to prove Conjecture 4, it suffices to prove it for λ of the form $\{1 \star k_1, 2 \star k_2\}$.

Theorem 4 below shows that Conjecture 4 holds for any λ of the form $\{1 \star k_1, 3 \star k_2\}$.

Theorem 4. *For each integer $a \geq 0$, there exists an integer $t_2 = t_2(a)$ such that for every integer $t \geq t_2$, there exists a K_t-minor-free graph that is not $\{1 \star (t-5a-9), 3 \star (2a+3)\}$-choosable.*

As $|\lambda|$ becomes very small compared to k_λ, say $|\lambda|$ is constant and k_λ tends to infinity, then λ-choosability becomes very close to k_λ-choosability. The following result, which generalizes the main result of Steiner [22], deals with such λ.

Theorem 5. *For every $\varepsilon \in (0,1)$ and $q \in \mathbb{N}$, there exists an integer $t_3 = t_3(q,\varepsilon)$ such that for every integer $t \geq t_3$ and $k_1, k_2, \dots, k_q \in \mathbb{N}$ satisfying*

$$\sum_{j=1}^{q} k_j \leq (2-\varepsilon)t,$$

there exists a K_t-minor-free graph G that is not $\{k_1, k_2, \dots, k_q\}$-choosable.

The $q = 1$ case of above theorem was proved by Steiner [22].

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