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Random perfect matchings in regular graphs

(Extended abstract)

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Abstract

We prove that in all regular robust expanders G, every edge is asymptotically equally likely contained in a uniformly chosen perfect matching M. We also show that given any fixed matching or spanning regular graph N in G, the random variable $|M \cap E(N)|$ is approximately Poisson distributed. This in particular confirms a conjecture and a question due to Spiro and Surya, and complements results due to Kahn and Kim who proved that in a regular graph every vertex is asymptotically equally likely contained in a uniformly chosen matching. Our proofs rely on the switching method and the fact that simple random walks mix rapidly in robust expanders.

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1 Introduction

A remarkable result due to Kahn and Kim [5] says that in *any* d-regular graph G, the probability that a vertex is contained in a uniformly chosen matching in G is $1-(1+o_d(1))d^{-\frac{1}{2}}$. This shows that the structure of a d-regular graph has essentially no impact on the probability that a vertex is contained in a uniformly chosen matching.

In this paper we are interested in uniformly chosen *perfect* matchings. Then, surely, each vertex is contained in every perfect matching. Hence, as the statement for vertices is

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trivial, what about the probability that an edge is contained in a random perfect matching? Is each edge equally likely contained in random perfect matching? A moment of thought reveals that this is wrong in a very strong sense. In every odd-regular graph with exactly one bridge, the bridge is contained in every perfect matching, while the edges adjacent to the bridge are contained in none of the perfect matchings. Therefore, in order to avoid a trivial statement further conditions are needed.

Hall's condition for the existence of perfect matchings in bipartite graphs says that the neighbourhood of an (independent) set should be at least as large as the set itself, which is clearly also a necessary condition. Here, we assume that this property is present in a robust sense in order to avoid the trivial scenarios mentioned above. More precisely, let $\nu, \tau > 0$ and G be a graph on n vertices. Then, we define the *ν-robust neighbourhood* $RN_{\nu,G}(S)$ of a set $S \subseteq V(G)$ in G to be the set of vertices of G which have at least νn neighbours in S. We say that G is a *robust (ν, τ)-expander* if $RN_{\nu,G}(S) \geq |S| + \nu n$ for each $S \subseteq V(G)$ satisfying $\tau n \leq |S| \leq (1-\tau)n$. Robust expansion is a fairly mild assumption and consequently it proved to be useful in several situations, see for example [3, 6, 7].

We denote by $\mathcal{P}(G)$ the set of all perfect matchings in G and write $M \sim U(\mathcal{P}(G))$ to refer to a uniformly chosen matching from $\mathcal{P}(G)$. Our main result implies that such matchings M are extremely well-distributed in robust expanders.

Theorem 1. *For any $\delta > 0$, there exists $\tau > 0$ such that for all $\nu > 0$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let $n \geq n_0$ be even and $d \geq \delta n$. Then, for any d-regular robust (ν, τ)-expander G on n vertices, $M \sim U(\mathcal{P}(G))$, and $e \in E(G)$, we have*

$$\mathbb{P}[e \in M] = (1 + o_n(1))d^{-1}.$$

In fact much more is true. Fix any matching N in G, let $M \sim U(\mathcal{P}(G))$, and consider $X := |M \cap N|$. Then, linearity of expectation and Theorem 1 imply that $\mathbb{E}[X] = (1 + o_n(1))d^{-1}|N|$. Employing the heuristic that each edge is *independently* present in $M \sim U(\mathcal{P}(G))$ with probability d^{-1}, then we expect that X has a binomial distribution with parameters $|N|$ and d^{-1}. This is approximated by a Poisson distribution with parameter $d^{-1}|N|$, whenever $|N|$ grows with n. Our next result confirms this.

To this end, we define the *total variation distance* of two integer-valued random variables Y and Z as $d_{\mathrm{TV}}(Y, Z) := \frac{1}{2}\sum_{k\in\mathbb{Z}} |\mathbb{P}[Y = k] - \mathbb{P}[Z = k]|$, which measures how close two distributions are. Moreover, we write $Y \sim \mathrm{Po}(\lambda)$ if Y is a random variable which follows a Poisson distribution with parameter λ.

Theorem 2. *For any $\delta > 0$, there exists $\tau > 0$ such that for all $\nu > 0$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let $n \geq n_0$ be even and $d \geq \delta n$. Then, for any d-regular robust (ν, τ)-expander G on n vertices, $M \sim U(\mathcal{P}(G))$, any matching N in G, $X := |M \cap N|$, and $Y \sim \mathrm{Po}(d^{-1}|N|)$, we have $d_{\mathrm{TV}}(X, Y) = o_n(1)$.*

The fact that N is a matching is not crucial for our argument, however note for example that if N is a star, then X is a $\{0, 1\}$-valued random variable. Hence, X can only converge to a Poisson distribution if N is somewhat spread out. In particular, when N is a spanning

r-regular graph for some fixed r, we can derive an analogue of Theorem 2 (see Section 2), which answers a question of Spiro and Surya [8].

Theorem 2 has some interesting consequences. We define $\mathrm{pm}(G) := |\mathcal{P}(G)|$ and suppose G and M are as in Theorem 2. Let N be a perfect matching in G. Then, Theorem 2 implies that

$$\frac{\mathrm{pm}(G-N)}{\mathrm{pm}(G)} = \mathbb{P}[M \cap N = \emptyset] = (1+o_n(1))e^{-\frac{n}{2d}}.$$

For a graph G with a perfect matching, we denote by G° a subgraph of G where one perfect matching is removed. Various combinatorial problems can be expressed as determining $\frac{\mathrm{pm}(G^\circ)}{\mathrm{pm}(G)}$. For example, when $G = K_{\frac{n}{2},\frac{n}{2}}$, this ratio is equal to the probability that a random permutation of order $\frac{n}{2}$ is fixed-point-free, and it is well known that this probability equals $(1+o_n(1))e^{-1}$. The case when $G = K_n$ also has a combinatorial interpretation, see [4].

Let $K_{a\times b}$ denote the complete multipartite graph with a parts, each of size b. As an interpolation between the cases $K_{\frac{n}{2},\frac{n}{2}}$ and K_n, one may ask whether $\mathrm{pm}(K^\circ_{r\times\frac{n}{r}})(\mathrm{pm}(K_{r\times\frac{n}{r}}))^{-1}$ converges to a limit. Johnston, Kayll, and Palmer [4] formulated this as a conjecture (and conjectured the limit value). Recently this was resolved by Spiro and Surya [8]. As all these graphs are robust expanders (excluding $K_{\frac{n}{2},\frac{n}{2}}$; we discuss bipartite graphs in Section 2), Theorem 2 reproves the result due to Spiro and Surya [8].

In fact, Spiro and Surya [8] also speculate whether for any $\alpha > \frac{1}{2}$, all regular graphs G on an even number n of vertices with $\delta(G) \geq \alpha n$ satisfy $\frac{\mathrm{pm}(G^\circ)}{\mathrm{pm}(G)} \to e^{-\frac{1}{2\alpha}}$, but consider this statement far too strong to be true. As it is trivial to show that graphs on n vertices with $\delta(G) \geq (\frac{1}{2}+o_n(1))n$ are robust expanders, Theorem 2 shows that this statement is actually true.

Our proof strategy is as follows (see the full version of this article [2] for more details). Let G, M, N, and X be as in the statement of Theorem 2. We estimate the ratios of the form $\frac{\mathbb{P}[X=k]}{\mathbb{P}[X=k-1]}$ via the so-called switching method. Knowing all relevant fractions of this type already exhibits the distribution of X, which has the advantage that the probabilities $\mathbb{P}[X=k]$ do not need to be calculated directly.

The switching method is implemented as follows. Fix a positive integer k and denote by $\mathcal{M}_k$ and $\mathcal{M}_{k-1}$ the sets of perfect matchings in G which contain precisely k and $k-1$ edges of N, respectively. Then, construct an auxiliary bipartite graph H on vertex classes $\mathcal{M}_k$ and $\mathcal{M}_{k-1}$ by joining two perfect matchings $M \in \mathcal{M}_k$ and $M' \in \mathcal{M}_{k-1}$ if there is a cycle C of length 2ℓ in G which contains precisely one edge of N and alternates between edges of M and M'. (In other words, $M \in \mathcal{M}_k$ and $M' \in \mathcal{M}_{k-1}$ are adjacent in H if $N \cap M' \subseteq N \cap M$ and the extra edge in $(N \cap M) \setminus M'$ can be 'switched out' of M to obtain M' by exchanging ℓ edges of M for ℓ edges of M', where these 2ℓ edges altogether form a cycle.)

Note that if all perfect matchings in $\mathcal{M}_k$ have degree (roughly) d_k in H, while all perfect matchings in $\mathcal{M}_{k-1}$ have degree (roughly) d_{k-1}, then $d_k|\mathcal{M}_k| \approx e(H) \approx d_{k-1}|\mathcal{M}_{k-1}|$. Hence, $\frac{\mathbb{P}[X=k]}{\mathbb{P}[X=k-1]} = \frac{|\mathcal{M}_k|}{|\mathcal{M}_{k-1}|} \approx \frac{d_{k-1}}{d_k}$. Therefore, the crux of the proof consists in precisely estimating the number of such alternating cycles.

Counting the number of cycles of a certain length can be achieved using random walks as follows. Given a d-regular graph, note that the number of walks of length ℓ starting at u is precisely d^ℓ, and so the probability that a simple random walk that starts in u is in v after ℓ steps is equal to the number of walks from u to v of length ℓ divided by d^ℓ. Since simple random walks are rapidly mixing in robust expanders, one can precisely estimate such probabilities, and therefore the number of such walks. A simple counting argument can eliminate those walks which are not paths, and so we can accurately count the number of cycles of fixed length in a regular robust expander. In practice, we have to consider simple random walks that use in every second step an edge from a fixed perfect matching M. However, this additional technicality does not affect the mixing properties of such walks and so we can still precisely count them.

We remark that Spiro and Surya [8] also used the switching method, which is common for this type of problems. Our contribution is to use longer cycles and perform the analysis with Markov chains; although the intuition is that the estimations become less precise with larger cycles, we employ key properties of Markov chains to show that in fact the opposite is true. Besides the fact that our results are substantially more general, the analysis also becomes significantly shorter and cleaner.

2 Extensions

In the full version of this paper we showed that uniformly chosen perfect matchings in robust expanders contain each edge asymptotically equally likely. In fact, for a larger set of disjoint edges, these events are approximately independent. As robust expanders are a fairly large class of graphs, this in particular contains graphs G on n vertices with $\delta(G) \geq (\frac{1}{2} + o_n(1))n$, which confirms a question of Spiro and Surya [8] in a strong form.

2.1 Regular subgraphs

Spiro and Surya [8] also suggest to estimate the probability that a uniformly chosen perfect matching of Turán graphs intersects a fixed spanning r-regular subgraph.

Theorem 3. *For any $\delta > 0$, there exists $\tau > 0$ such that for all $\nu > 0$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let $n \geq n_0$ be even, $d \geq \delta n$, and let $r \leq n^{\frac{1}{50}}$ be a positive integer. Then, for any d-regular robust (ν, τ)-expander G on n vertices, $M \sim U(\mathcal{P}(G))$, any spanning r-regular subgraph N in G, $X \coloneqq |M \cap E(N)|$, and $Y \sim \mathrm{Po}(\frac{rn}{2d})$, we have $d_{\mathrm{TV}}(X, Y) = o_n(1)$.*

As a corollary, one can calculate the probability that r perfect matchings, each chosen independently and uniformly at random, are (edge-)disjoint. This relates to a problem of Ferber, Hänni, and Jain [1], which asks for the probability of selecting r edge-disjoint copies of a graph H in a host graph G. They answer this question for Hamilton cycles in the complete graph. The following corollary is an analogue for perfect matchings in the more general class of robust expanders. The proof follows immediately from Theorem 3 by induction on r.

Corollary 4. *For any $\delta > 0$, there exists $\tau > 0$ such that for all $\nu > 0$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let $n \geq n_0$ be even, $d \geq \delta n$, and $r \leq n^{\frac{1}{50}}$. Then, for any d-regular robust (ν, τ)-expander G on n vertices and independent $M_1, \dots, M_r \sim U(\mathcal{P}(G))$, we have*

$$\mathbb{P}[M_1, \dots, M_r \text{ are disjoint}] = (1 + o_n(1))e^{-\frac{n}{2d}\binom{r}{2}}.$$

2.2 Bipartite graphs

Of particular interest are perfect matchings in (balanced) bipartite graphs, but bipartite graphs are not robust expanders as the neighbourhood of one of the partition classes is only at most as large as the class itself. However, the notion of robust expanders can be adapted to bipartite graphs. Let G be a bipartite graph with vertex partition (A, B) and $|A| = |B| = n$. We say that G is a *bipartite robust (ν, τ)-expander* if $RN_{\nu,G}(S) \geq |S| + \nu n$ for each $S \subseteq A$ satisfying $\tau n \leq |S| \leq (1 - \tau)n$.

The following is an analogue of Theorems 1–3 for bipartite graphs. This then also includes an approximation for the number of derangements.

Theorem 5. *For any $\delta > 0$, there exists $\tau > 0$ such that for all $\nu > 0$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let $n \geq n_0$, $d \geq \delta n$, and $r \leq n^{\frac{1}{50}}$. Let G be a balanced bipartite d-regular robust (ν, τ)-expander on $2n$ vertices and suppose that N is a matching in G or a spanning r-regular subgraph of G. Let $M \sim U(\mathcal{P}(G))$, let $X := |M \cap E(N)|$, let $Y \sim \mathrm{Po}(d^{-1}e(N))$, and let $e \in E(G)$. Then, $\mathbb{P}[e \in M] = (1 + o_n(1))d^{-1}$ and $d_{\mathrm{TV}}(X, Y) = o_n(1)$.*

References

[1] A. Ferber, K. Hänni, and V. Jain. The probability of selecting k edge-disjoint Hamilton cycles in the complete graph. *arXiv:2001.01149*, 2020.

[2] B. Granet and F. Joos. Random perfect matchings in regular graphs. *arXiv:2301.10131*, 2023.

[3] V. Gruslys and S. Letzter. Cycle partitions of regular graphs. *Combin. Probab. Comput.*, 30:526–549, 2021.

[4] D. Johnston, P. M. Kayll, and C. Palmer. Deranged matchings: proofs and conjectures. *arXiv:2209.11319*, 2022.

[5] J. Kahn and J. H. Kim. Random matchings in regular graphs. *Combinatorica*, 18:201–226, 1998.

[6] D. Kühn, A. Lo, D. Osthus, and K. Staden. The robust component structure of dense regular graphs and applications. *Proc. Lond. Math. Soc.*, 110:19–56, 2015.

[7] D. Kühn and D. Osthus. Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments. *Adv. Math.*, 237:62–146, 2013.

[8] S. Spiro and E. Surya. Counting deranged matchings. *arXiv:2211.01872*, 2022.