Hamilton cycles in pseudorandom graphs

(Extended abstract)

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Abstract

Finding general conditions which ensure that a graph is Hamiltonian is a central topic in graph theory. An old and well known conjecture in the area states that any d-regular n-vertex graph G whose second largest eigenvalue in absolute value $\lambda(G)$ is at most d/C, for some universal constant $C > 0$, has a Hamilton cycle. We obtain two main results which make substantial progress towards this problem. Firstly, we settle this conjecture in full when the degree d is at least a small power of n. Secondly, in the general case we show that $\lambda(G) \leq d/C(\log n)^{1/3}$ implies the existence of a Hamilton cycle, improving the 20-year old bound of $d/\log^{1-o(1)} n$ of Krivelevich and Sudakov. We use in a novel way a variety of methods, such as a robust Pósa rotation-extension technique, the Friedman-Pippenger tree embedding with rollbacks and the absorbing method, combined with additional tools and ideas.

Our results have several interesting applications, giving best bounds on the number of generators which guarantee the Hamiltonicity of random Cayley graphs, which is an important partial case of the well known Hamiltonicity conjecture of Lovász. They can also be used to improve a result of Alon and Bourgain on additive patterns in multiplicative subgroups.

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1 Introduction

A *Hamilton cycle* in a graph G is a cycle passing through all the vertices of G. If it exists, then G is called *Hamiltonian*. Being one of the most central notions in Graph Theory, it has been extensively studied by numerous researchers, see e.g., [1, 9, 13, 15, 16, 20, 23, 28, 31, 32, 34, 38], and the surveys [22, 33]. In particular, the problem of deciding Hamiltonicity of a graph is known to be NP-complete and thus, finding general conditions which ensure that G has a Hamilton cycle is one of the most popular topics in Graph Theory. For instance, two famous theorems of this nature are the celebrated result of Dirac [19], which states that if the minimum degree of an n-vertex graph G is at least $n/2$, then G contains a Hamilton cycle, and the criterion of Chvátal and Erdős [13] that a graph is Hamiltonian if its connectivity number is at least as large as its independence number.

In fact, most of the classical criteria for Hamiltonicity focus on rather dense graphs. A prime example of this is clearly Dirac's theorem stated above, but also the Chvátal-Erdős condition requires the graph to be relatively dense, of average degree $\Omega(\sqrt{n})$. In contrast, sufficient conditions that ensure Hamiltonicity of sparse graphs seem much more difficult to obtain. A natural starting point towards this topic is to consider sparse random graphs, to which a lot of research has been dedicated in the last 50 years. In a breakthrough paper in 1976, Pósa [38] proved that the binomial random graph model $G(n, p)$ with $p \geq C \log n / n$ for some large constant C almost surely contains a Hamilton cycle. In doing so, he invented the influential *rotation-extension technique* for finding long cycles and paths, which has found numerous further applications since then. Pósa's result was later refined by Korshunov [26] and in 1983, a more precise threshold for Hamiltonicity was obtained by Bollobás [8] and Komlós and Szemerédi [25], who independently showed that if $p = (\log n + \log \log n + \omega(1))/n$, then $G(n, p)$ is almost surely Hamiltonian. It is a standard exercise to note that this is essentially tight - indeed, if $p = (\log n + \log \log n - \omega(1))/n$, then $G(n, p)$ almost surely has a vertex with degree at most 1, and hence is not Hamiltonian. In parallel, significant attention has also been given to the Hamiltonicity of the random d-regular graph model $G_{n,d}$ - it is known that $G_{n,d}$ almost surely contains a Hamilton cycle for all values of $3 \leq d \leq n - 1$. For this result, the reader is referred to Cooper, Frieze and Reed [14] and Krivelevich, Sudakov, Vu and Wormald [30] and their references.

Given the success of the study of Hamilton cycles in sparse random graphs, it became natural to then consider pseudorandom graphs, which are deterministic graphs that resemble random graphs in various important properties. A convenient way to express pseudorandomness is via spectral techniques and was introduced by Alon. An (n, d, λ)-*graph* is an n-vertex d-regular graph G whose second largest eigenvalue in absolute value, $\lambda(G)$, is such that $\lambda(G) \leq \lambda$. Roughly speaking, $\lambda(G)$ is a measure of how "smooth" the edge-distribution of G is, and the smaller its value, the closer to "random" G behaves. The reader is referred to [29] for a detailed survey concerning pseudorandom graphs.

In a rather influential paper, Krivelevich and Sudakov [27] employed Pósa's rotation-extension technique to prove the very general result that (n, d, λ)-graphs are Hamiltonian, provided λ is significantly smaller than d. Precisely, they showed that if n is sufficiently

large, then

$$d/\lambda \geq \frac{1000 \log n (\log \log \log n)}{(\log \log n)^2} \tag{1}$$

guarantees that any (n, d, λ)-graph contains a Hamilton cycle. It is worth mentioning that Hefetz, Krivelevich and Szabó [23] provided a more general sufficient condition for Hamiltonicity in terms of expansion and some variant of high connectivity, yet for (n, d, λ)-graphs their condition essentially reduces to (1).

The above result on Hamiltonicity of (n, d, λ)-graphs has found numerous applications in the last 20 years towards some well-known problems, some of which we will discuss later. Given its significance and generality, it leads to the very natural and fundamental question of whether a smaller ratio of d/λ is already sufficient to imply Hamiltonicity. Krivelevich and Sudakov [27] conjectured that it should suffice that d/λ is only a large enough constant.

Conjecture 1.1. *There exists an absolute constant $C > 0$ such that any (n, d, λ)-graph with $d/\lambda \geq C$ contains a Hamilton cycle.*

2 Main results

Despite the plethora of incentives, there has been no improvement until now on the Krivelevich and Sudakov bound stated in (1). We make significant progress towards Conjecture 1.1 in two ways. First, we improve on the Krivelevich and Sudakov bound in general by showing that a spectral ratio of order $(\log n)^{1/3}$ already guarantees Hamiltonicity.

Theorem 2.1. *There exists a constant $C > 0$ such that any (n, d, λ)-graph with $d/\lambda \geq C(\log n)^{1/3}$ contains a Hamilton cycle.*

The proof of the above result will rely on the Pósa rotation-extension method with various new ideas. Namely, we will need to develop some techniques in order to use this method in a *robust* manner.

Secondly, we confirm Conjecture 1.1 in full when the degree is polynomial in the order of the graph.

Theorem 2.2. *For every constant $\alpha > 0$, there exists a constant $C > 0$ such that any (n, d, λ)-graph with $d \geq n^\alpha$ and $d/\lambda \geq C$ contains a Hamilton cycle.*

In fact, Theorem 2.2 is a corollary of a more general statement that we will prove which in particular states that (n, d, λ)-graphs with linearly many vertex-disjoint cycles are Hamiltonian.

3 Applications and related problems

Both Theorem 2.1 and Theorem 2.2 immediately yield improvements in several applications which made use of the result of Krivelevich and Sudakov. One application is an

important special case of a famous open question of Lovász [35] from 1969 concerning the Hamiltonicity of a certain class of well-behaved graphs (see e.g., [17] and its references for more background on the problem).

Conjecture 3.1. *Every connected vertex-transitive graph contains a Hamilton path, and, except for five known examples, a Hamilton cycle.*

Since Cayley graphs are vertex-transitive and none of the five known exceptions in Lovász's conjecture is a Cayley graph, the conjecture in particular includes the following, which was asked much earlier in 1959 by Rapaport Strasser [39].

Conjecture 3.2. *Every connected Cayley graph is Hamiltonian.*

For these conjectures, a proof is currently out of sight. Indeed, notable progress towards them in their full generality are a result of Babai [5] that every vertex-transitive n-vertex graph contains a cycle of length $\Omega(\sqrt{n})$ (see [18] for a recent improvement) and a result of Christofides, Hladký and Máthé [12] that every vertex-transitive graph of linear minimum degree contains a Hamilton cycle.

Given this, it is natural to consider the "random" version of Conjecture 3.2. Indeed, Alon and Roichman [4] showed that in any group G, a random set S of $O(\log |G|)$ elements is such that the Cayley graph generated by them, $\Gamma(G, S)$, is almost surely connected. Therefore, a particular instance of Conjecture 3.2 is to show that $\Gamma(G, S)$ is almost surely Hamiltonian, which is itself a conjecture of Pak and Radoičić [37]. In fact, this relates directly to Conjecture 1.1 since it can be shown, generalizing the result of Alon and Roichman, that if $|S| \geq C \log |G|$ for some large constant C, then $\Gamma(G, S)$ is almost surely an (n, d, λ)-graph with $d/\lambda \geq K$ for some large constant K. Hence, Conjecture 1.1 would imply the Hamiltonicity of $\Gamma(G, S)$. Improving on several earlier results [11, 27, 36] we will show how Theorem 2.1 can be used to prove that if $|S|$ is of order $\log^{5/3} n$, then $\Gamma(G, S)$ is almost surely Hamiltonian. We will also give an improved bound on a related problem of Akbari, Etesami, Mahini, and Mahmoody [3] concerning Hamilton cycles in coloured complete graphs which use only few colours.

Another application of our results concerns one of the central themes in Additive Combinatorics, the interplay between the two operations sum and product. A well-known fact in this area is that any multiplicative subgroup A of the finite field $\mathbb{F}_q$ of size at least $q^{3/4}$ must contain two elements x, y such that $x + y$ also belongs to A. Motivated by this, Alon and Bourgain [3] studied more complex additive structures in multiplicative subgroups. In particular, they proved that when a subgroup has size $|A| \geq q^{3/4}(\log q)^{1/2-o(1)}$, then there is a cyclic ordering of the elements of A such that the sum of any two consecutive elements is also in A. Using Theorem 2.2, we can improve on Alon and Bourgain's result, showing that the additional polylog-factor can be avoided. This shows that when $|A|$ is of order $q^{3/4}$, not only does it contain $x, y, x + y \in A$ but also much more complex structures.

Finally, we give an application of our techniques to another problem related to Conjecture 3.2. Motivated by this conjecture, Pak and Radoičić [37] showed that every group G has a set of generators S of size at most $\log_2 |G|$ such that the Cayley graph $\Gamma(G, S)$

is Hamiltonian, which is optimal since there are groups that do not have generating sets of size smaller than $\log_2 |G|$. Since their proof relies on the classification of finite simple groups, they asked to find a classification-free proof of this result. Using the methods developed for the proof of Theorem 2.2 we give a classification-free proof that there is always such a set S with $|S| = O(\log n)$.

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