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The Minimum Degree Removal Lemma Thresholds

(Extended abstract)

Lior Gishboliner* Zhihan Jin* Benny Sudakov*

Abstract

The graph removal lemma is a fundamental result in extremal graph theory which says that for every fixed graph H and $\varepsilon > 0$, if an n-vertex graph G contains εn^2 edge-disjoint copies of H then G contains $\delta n^{v(H)}$ copies of H for some $\delta = \delta(\varepsilon, H) > 0$. The current proofs of the removal lemma give only very weak bounds on $\delta(\varepsilon, H)$, and it is also known that $\delta(\varepsilon, H)$ is not polynomial in ε unless H is bipartite. Recently, Fox and Wigderson initiated the study of minimum degree conditions guaranteeing that $\delta(\varepsilon, H)$ depends polynomially or linearly on ε. We answer several questions of Fox and Wigderson on this topic.

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1 Introduction

The graph removal lemma, first proved by Ruzsa and Szemerédi [22], is a fundamental result in extremal graph theory. It also has important applications to additive combinatorics and property testing. The lemma states that for every fixed graph H and $\varepsilon > 0$, if an n-vertex graph G contains εn^2 edge-disjoint copies of H then G it contains $\delta n^{v(H)}$ copies of H, where $\delta = \delta(\varepsilon, H) > 0$. Unfortunately, the current proofs of the graph removal lemma give only very weak bounds on $\delta = \delta(\varepsilon, H)$ and it is a very important problem to understand the dependence of δ on ε. The best known result, due to Fox [11], proves that $1/\delta$ is at most a tower of exponents of height logarithmic in $1/\varepsilon$. Ideally, one would like to have better

*Department of Mathematics, ETH, Zürich, Switzerland. Research supported in part by SNSF grant 200021_196965. Email: **{lior.gishboliner, zhihan.jin, benjamin.sudakov}@math.ethz.ch**.

bounds on $1/\delta$, where an optimal bound would be that δ is polynomial in ε. However, it is known [2] that $\delta(\varepsilon, H)$ is polynomial in ε only if H is bipartite. This situation led Fox and Wigderson [12] to initiate the study of minimum degree conditions which guarantee that $\delta(\varepsilon, H)$ depends polynomially or linearly on ε. Formally, let $\delta(\varepsilon, H; \gamma)$ be the maximum $\delta \in [0, 1]$ such that if G is an n-vertex graph with minimum degree at least γn and with εn^2 edge-disjoint copies of H, then G contains $\delta n^{v(H)}$ copies of H.

Definition 1.1. *Let H be a graph.*

1. *The* linear removal threshold *of H, denoted $\delta_{\text{lin-rem}}(H)$, is the infimum γ such that $\delta(\varepsilon, H; \gamma)$ depends linearly on ε, i.e. $\delta(\varepsilon, H; \gamma) \geq \mu\varepsilon$ for some $\mu = \mu(\gamma) > 0$ and all $\varepsilon > 0$.*

2. *The* polynomial removal threshold *of H, denoted $\delta_{\text{poly-rem}}(H)$, is the infimum γ such that $\delta(\varepsilon, H; \gamma)$ depends polynomially on ε, i.e. $\delta(\varepsilon, H; \gamma) \geq \mu\varepsilon^{1/\mu}$ for some $\mu = \mu(\gamma) > 0$ and all $\varepsilon > 0$.*

Trivially, $\delta_{\text{lin-rem}}(H) \geq \delta_{\text{poly-rem}}(H)$. Fox and Wigderson [12] initiated the study of $\delta_{\text{lin-rem}}(H)$ and $\delta_{\text{poly-rem}}(H)$, and proved that $\delta_{\text{lin-rem}}(K_r) = \delta_{\text{poly-rem}}(K_r) = \frac{2r-5}{2r-3}$ for every $r \geq 3$, where K_r is the clique on r vertices. They further asked to determine the removal lemma thresholds of odd cycles. Here we completely resolve this question. The following theorem handles the polynomial removal threshold.

Theorem 1.2. $\delta_{\text{poly-rem}}(C_{2k+1}) = \frac{1}{2k+1}$.

Theorem 1.2 also answers another question of Fox and Wigderson [12], of whether $\delta_{\text{lin-rem}}(H)$ and $\delta_{\text{poly-rem}}(H)$ can only obtain finitely many values on r-chromatic graphs H for a given $r \geq 3$. Theorem 1.2 shows that $\delta_{\text{poly-rem}}(H)$ obtains infinitely many values for 3-chromatic graphs. In contrast, $\delta_{\text{lin-rem}}(H)$ obtains only three possible values for 3-chromatic graphs. Indeed, the following theorem determines $\delta_{\text{lin-rem}}(H)$ for every 3-chromatic H. An edge xy of H is called *critical* if $\chi(H - xy) < \chi(H)$.

Theorem 1.3. *For a graph H with $\chi(H) = 3$, it holds that*

$$\delta_{\text{lin-rem}}(H) = \begin{cases} \frac{1}{2} & H \text{ has no critical edge,} \\ \frac{1}{3} & H \text{ has a critical edge and contains a triangle,} \\ \frac{1}{4} & H \text{ has a critical edge and } \text{odd-girth}(H) \geq 5. \end{cases}$$

Theorems 1.2 and 1.3 show a separation between the polynomial and linear removal thresholds, giving a sequence of graphs (i.e. $C_5, C_7, \ldots$) where the polynomial threshold tends to 0 while the linear threshold is constant $\frac{1}{4}$. The proof of Theorem 1.3 appears in the full version of this paper.

The parameters $\delta_{\text{poly-rem}}$ and $\delta_{\text{lin-rem}}$ are related to two other well-studied minimum degree thresholds: the chromatic threshold and the homomorphism threshold. The chromatic threshold of a graph H is the infimum γ such that every n-vertex H-free graph G with

$\delta(G) \geq \gamma n$ has bounded cromatic number, i.e., there exists $C = C(\gamma)$ such that $\chi(G) \leq C$. The study of the chromatic threshold originates in the work of Erdős and Simonovits [10] from the '70s. Following multiple works [4, 14, 15, 7, 5, 24, 25, 18, 6, 13, 19], the chromatic threshold of every graph was determined by Allen et al. [1].

Moving on to the homomorphism threshold, we define it more generally for families of graphs. The *homomorphism threshold* of a graph-family $\mathcal{H}$, denoted $\delta_{\text{hom}}(\mathcal{H})$, is the infimum γ for which there exists an $\mathcal{H}$-free graph $F = F(\gamma)$ such that every n-vertex $\mathcal{H}$-free graph G with $\delta(G) \geq \gamma n$ is homomorphic to F. When $\mathcal{H} = \{H\}$, we write $\delta_{\text{hom}}(H)$. This parameter was widely studied in recent years [17, 21, 16, 8, 23]. It turns out that δ_{hom} is closely related to $\delta_{\text{poly-rem}}(H)$, as the following theorem shows. For a graph H, let $\mathcal{I}_H$ denote the set of all minimal (with respect to inclusion) graphs H' such that H is homomorphic to H'.

Theorem 1.4. *For every graph H, $\delta_{\text{poly-rem}}(H) \leq \delta_{\text{hom}}(\mathcal{I}_H)$.*

Note that $\mathcal{I}_{C_{2k+1}} = \{C_3, C_5, \dots, C_{2k+1}\}$. Using this, the upper bound in Theorem 1.2 follows immediately by combining Theorem 1.4 with the result of Ebsen and Schacht [8] that $\delta_{\text{hom}}(\{C_3, C_5, \dots, C_{2k+1}\}) = \frac{1}{2k+1}$. The lower bound in Theorem 1.2 was established in [12].

2 Proof of Theorem 1.4

We say that an n-vertex graph G is ε-*far* from a graph property $\mathcal{P}$ (e.g. being H-free for a given graph H, or being homomorphic to a given graph F) if one must delete at least εn^2 edges to make G satisfy $\mathcal{P}$. Trivially, if G has εn^2 edge-disjoint copies of H, then it is ε-far from being H-free. The following result is from [20].

Theorem 2.1. *For every graph F on f vertices and for every $\varepsilon > 0$, there is $q = q_F(\varepsilon) =$ poly(f/ε), such that the following holds. If a graph G is ε-far from being homomorphic to F, then for a sample of q vertices $x_1, \dots, x_q \in V(G)$, taken uniformly with repetitions, it holds that $G[\{x_1, \dots, x_q\}]$ is not homomorphic to F with probability at least $\frac{2}{3}$.*

Theorem 2.1 is proved in Section 2 of [20]. In fact, [20] proves a more general result on property testing of the so-called 0/1-partition properties. Such a property is given by an integer f and a function $d : [f]^2 \to \{0, 1, \perp\}$, and a graph G satisfies the property if it has a partition $V(G) = V_1 \cup \cdots \cup V_f$ such that for every $1 \leq i, j \leq f$ (possibly $i = j$), it holds that (V_i, V_j) is complete if $d(i, j) = 1$ and (V_i, V_j) is empty if $d(i, j) = 0$ (if $d(i, j) = \perp$ then there are no restrictions). One can express the property of having a homomorphism into F in this language, simply by setting $d(i, j) = 0$ for $i = j$ and $ij \notin E(F)$. In [20], the class of these partition properties is denoted $\mathcal{GPP}_{0,1}$, and every such property is shown to be testable by sampling poly(f/ε) vertices. This implies Theorem 2.1.

For a graph H on $[h]$ and integers $s_1, s_2, \dots, s_h > 0$, we denote by $H[s_1, \dots, s_h]$ the blow-up of H where each vertex $i \in V(H)$ is replaced by a set S_i of size s_i. The following lemma is standard, and follows from the hypergraph version of the Kővári-Sós-Turán theorem [9].

Lemma 2.2. *Let H be a fixed graph on vertex set $[h]$ and let $s_1, s_2, \dots, s_h \in \mathbb{N}$. There exists a constant $c = c(H, s_1, \dots, s_h) > 0$ such that the following holds. Let G be an n-vertex graph and $V_1, \dots, V_h \subseteq V(G)$. Suppose that G contains at least ρn^h copies of H mapping i to V_i for all $i \in [h]$. Then G contains at least $c\rho^{\frac{1}{c}} \cdot n^{s_1+\dots+s_h}$ copies of $H[s_1, \dots, s_h]$ mapping S_i to V_i for all $i \in [h]$.*

Proof of Theorem 1.4. Recall that $\mathcal{I}_H$ is the set of minimal graphs H' (with respect to inclusion) such that H is homomorphic to H'. For convenience, put $\delta := \delta_{\text{hom}}(\mathcal{I}_H)$. Our goal is to show that $\delta_{\text{poly-rem}}(H) \leq \delta + \alpha$ for every $\alpha > 0$. So fix $\alpha > 0$ and let G be a graph with minimum degree $\delta(G) \geq (\delta + \alpha)n$ and with εn^2 edge-disjoint copies of H. By the definition of the homomorphism threshold, there is an $\mathcal{I}_H$-free graph F (depending only on $\mathcal{I}_H$ and α) such that if a graph G_0 is $\mathcal{I}_H$-free and has minimum degree at least $(\delta + \frac{\alpha}{2}) \cdot |V(G_0)|$, then G_0 is homomorphic to F. Observe that if a graph G_0 is homomorphic to F then G_0 is H-free, because F is free of any homomorphic image of H. It follows that G is ε-far from being homomorphic to F, because G is ε-far from being H-free. Now we apply Theorem 2.1. Let $q = q_F(\varepsilon)$ be given by Theorem 2.1. We assume that $q \gg \frac{\log(1/\alpha)}{\alpha^2}$ and $n \gg q^2$ without loss of generality. Sample q vertices $x_1, \dots, x_q \in V(G)$ with repetition and let $X = \{x_1, \dots, x_q\}$. By Theorem 2.1, $G[X]$ is not homomorphic to F with probability at least $2/3$. As $n \gg q^2$, the vertices $x_1, \dots, x_q$ are pairwise-distinct with probability at least 0.99. Also, for every $i \in [q]$, the number of indices $j \in [q] \setminus \{i\}$ with $x_i x_j \in E(G)$ dominates a binomial distribution $\mathrm{B}(q-1, \frac{\delta(G)}{n})$. By the Chernoff bound (see e.g. [3, Appendix A]) and as $\delta(G) \geq (\delta + \alpha)n$, the number of such indices is at least $(\delta + \frac{\alpha}{2})q$ with probability $1 - e^{-\Omega(q\alpha^2)}$. Taking the union bound over $i \in [q]$, we get that $\delta(G[X]) \geq (\delta + \frac{\alpha}{2})|X|$ with probability at least $1 - qe^{-\Omega(q\alpha^2)} \geq 0.9$, as $q \gg \frac{\log(1/\alpha)}{\alpha^2}$. Hence, with probability at least $\frac{1}{2}$ it holds that $\delta(G[X]) \geq (\delta + \frac{\alpha}{2})|X|$ and $G[X]$ is not homomorphic to F. If this happens, then $G[X]$ is not $\mathcal{I}_H$-free (by the choice of F), hence $G[X]$ contains a copy of some $H' \in \mathcal{I}_H$. By averaging, there is $H' \in \mathcal{I}_H$ such that $G[X]$ contains a copy of H' with probability at least $\frac{1}{2|\mathcal{I}_H|}$. Put $k = |V(H')|$ and let M be the number of copies of H' in G. The probability that $G[X]$ contains a copy of H' is at most $M(\frac{q}{n})^k$. Using the fact that $q = \text{poly}_{H,\alpha}(\frac{1}{\varepsilon})$, we conclude that $M \geq \frac{1}{2|\mathcal{I}_H|} \cdot (\frac{n}{q})^k \geq \text{poly}_{H,\alpha}(\varepsilon) n^k$. As $H \to H'$, there exists H'', a blow-up of H', such that H'' have the same number of vertices as H, and that $H \subset H''$. By Lemma 2.2 for H' with $V_i = V(G)$ for all i, there exist $\text{poly}_{H,\alpha}(\varepsilon) n^{v(H'')}$ copies of H'' in G, and thus $\text{poly}_{H,\alpha}(\varepsilon) n^{v(H)}$ copies of H. This completes the proof. □

3 Concluding remarks and open questions

It would be interesting to determine the possible values of $\delta_{\text{poly-rem}}(H)$ for 3-chromatic graphs H. So far we know that $\frac{1}{2k+1}$ is a value for each $k \geq 1$. Is there a graph H with $\frac{1}{5} < \delta_{\text{poly-rem}}(H) < \frac{1}{3}$? Also, is it true that $\delta_{\text{poly-rem}}(H) > \frac{1}{5}$ if H is not homomorphic to C_5?

Another question is whether the inequality in Theorem 1.4 is always tight, i.e. is it always true that $\delta_{\text{poly-rem}}(H) = \delta_{\text{hom}}(\mathcal{I}_H)$?

Finally, we wonder whether the parameters $\delta_{\text{poly-rem}}(H)$ and $\delta_{\text{lin-rem}}(H)$ are monotone, in the sense that they do not increase when passing to a subgraph of H. We are not aware of a way of proving this without finding $\delta_{\text{poly-rem}}(H), \delta_{\text{lin-rem}}(H)$.

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