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On the structure and values of betweenness centrality in dense betweenness-uniform graphs

(Extended abstract)

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Abstract

Betweenness centrality is a network centrality measure based on the amount of shortest paths passing through a given vertex. A graph is betweenness-uniform (BUG) if all vertices have an equal value of betweenness centrality. In this contribution, we focus on betweenness-uniform graphs with betweenness centrality below one. We disprove a conjecture about the existence of a BUG with betweenness value α for any rational number α from the interval $(3/4, \infty)$ by showing that only very few betweenness centrality values below $6/7$ are attained for at least one BUG. Furthermore, among graphs with diameter at least three, there are no betweenness-uniform graphs with a betweenness centrality smaller than one. In graphs of smaller diameter, the same can be shown under a uniformity condition on the components of the complement.

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1 Introduction and preliminaries

In network science, it is useful to have the ability to evaluate the nodes according to their importance in the given network with respect to some criterion. Centrality measures, a tool for this evaluation, are based on various properties of nodes such as the sizes of their neighborhoods or their distances to other parts of the network. The choice of the most suitable centrality heavily depends on what is being represented by the corresponding network, resulting in the impossibility to directly compare these measures.

In this work, we study the properties of betweenness centrality, which is based on the fraction of shortest paths passing through the given vertex. More precisely, for a vertex $u \in V(G)$ of a graph $G = (V, E)$ the *betweenness centrality* of u (or shortly just betweenness) is

$$B(u) = \sum_{\{v,w\} \in V(G)} \frac{\sigma_{vw}(u)}{\sigma_{vw}},$$

where σ_{vw} denotes the number of shortest paths between v and w and $\sigma_{vw}(u)$ is the number of such paths passing through u [4]. The betweenness centrality of a graph G, denoted $B(G)$, is defined as the average betweenness centrality of its vertices. Although having a wide range of applications, for example in the assessment of electric-grid vulnerability [1], in the measuring of dependencies in object-oriented software systems [10], in arms transfer [2] or in citation networks [12], there has not been many theoretical results about properties of betweenness centrality.

One of the few theoretical aspects studied is the distribution of betweenness centrality values and specifically its extremal cases, be it either distinct betweenness for all vertices [9] or betweenness centrality being the same for all vertices [6, 7, 8]. In the latter case, such graph G is called *betweenness-uniform*, or shortly a BUG, and its complement $\overline{G}$ is called a coBUG. Our main result is in understanding of the structure of betweenness-uniform graphs with betweenness value below one and in disproving the following conjecture.

Conjecture 1 (Coroničová-Hurajová, Madaras (2013) [6]). *For any rational value α in the interval $(3/4, \infty)$, there exists a betweenness-uniform graph with betweenness value of α.*

Note that the authors of the above-mentioned conjecture use a definition of betweenness that counts contributions of ordered pairs of vertices, whereas our definition counts contributions of unoriented pairs, resulting in betweenness values smaller by a factor of one-half.

In the following text, we use standard graph-theoretic notation, namely the *diameter* $\operatorname{diam}(G)$ being the longest of the shortest paths inside G, and $\overline{G} = (V, \binom{n}{2} \setminus E)$ being the complement of the graph G. A graph is *k-regular* if each vertex has exactly k neighbors. The fact that graphs G and H are isomorphic is denoted by $G \approx H$. Also, K_n denotes the complete graph on n vertices and $K_{a,b}$ denotes the complete bipartite graph with parts of sizes a and b.

2 Known constructions of betweenness-uniform graphs and their values

It is an open problem to characterize all betweenness-uniform graphs and the spectrum of betweenness values that are attained on their vertices. We introduce some of the known classes of BUGs to give the reader some intuition.

The simplest BUGs are the *vertex-transitive* graphs, i.e., the graphs in which for each pair of vertices there is an automorphism of the graph mapping one vertex onto the other. [6, 11] This holds for example for cycles, complete graphs, or complete bipartite graphs with the same sizes of parts.

It has been shown that the average betweenness centrality of an arbitrary connected graph G is related to the average pairwise distances of its vertices: $B(G) = \frac{1}{2}(n-1)(l(G)-1)$, where $l(G)$ is the average distance between pairs of vertices G and n is the number of its vertices [3, 5]. Using this expression and the fact that average betweenness is equal to betweenness of any vertex in a BUG, it is not hard to see that any half-integer can be obtained as a value of betweenness in a vertex-transitive graph.

Moreover, it has been shown that for a given n, there are superpolynomially many BUGs that are not vertex-transitive [6]. A class containing many such graphs is the class of distance-regular graphs. A graph is *distance-regular* if for any two vertices x and y, the number of vertices in distance a from x and in distance b from y depends only on the triple $(a, b, \mathrm{dist}(x, y))$.Any distance-regular graph is a BUG [6].

There is a construction allowing the creation of a BUG G' from a smaller k-regular BUG G. Let H be a disjoint union of ℓ distinct cliques of multiplicity n_i and order r_i for each $i \in \{1, \dots, \ell\}$. Then replace each vertex of G by H and for H_x and H_y being the copies of H that replaced x and y, make a full-join of H_x and H_y whenever $xy \in E(G)$ [6]. The betweenness value in G' is given by

$$B(G') = \frac{1}{2}\Big(mB(G) + m - 1 - \sum_{i=1}^{\ell} \frac{n_i r_i(r_i - 1)}{m}\Big),$$

where $m = |V(H)| = \sum_{i=1}^{\ell} n_i r_i$. Take $G \approx K_n$ for $n \geq 2$ and consider graphs G', G'' obtained by doing the above-mentioned construction with disjoint unions of cliques $H' = \bigcup_{a=1}^{c} K_{r'_a}$ and $H'' = \bigcup_{a=1}^{c-1} K_{r''_a}$ where for $i \leq c-2$, $r'_i = r''_i$ and $r''_{c-1} = r'_{c-1} + r'_c$. Then with an increasing number of vertices in the disjoint unions of cliques, we can construct G', G'' with decreasing $|B(G') - B(G'')|$.

3 Betweenness-uniform graphs with betweenness value below one

Based on the fact that only non-adjacent pairs of vertices can contribute to the betweenness of other vertices, we show the following relation between the density of the complement of G, $\overline{G}$, and the resulting average betweenness centrality.

Lemma 2. *For a connected graph G, we have $B(G) \geq \frac{|E(\overline{G})|}{|V(\overline{G})|}$, with equality if and only if G has diameter at most 2.*

3.1 No BUGs with diameter ≥ 3 and betweenness below one

By Lemma 2, betweenness-uniform graphs with betweenness-centrality below one are quite dense. For dense graphs, it is often easier to analyze their structure and properties by studying their complements. We start by observing that complements of graphs with diameter at least three have a simple structure.

Theorem 3. *A connected graph G satisfies diam$(G) \geq 3$ if and only if $\overline{G}$ is connected and contains a spanning tree which is a double star.*

By combining Theorem 3 and Lemma 2 and observing that a complement of a double star is not a BUG leads to the following corollary.

Corollary 4. *There is no connected betweenness-uniform graph of diameter greater than two having betweenness centrality below one.*

3.2 Structure of BUGs with diameter 2 and betweenness below one

Due to Corollary 4, we can restrict the remaining analysis to betweenness-uniform graphs with diameter two. In these graphs, each shortest path contributes to exactly one vertex.

Consider $\overline{G}$ a complement of a BUG of diameter two with $|V(\overline{G})| = n$. We say that a vertex v is *close* to an edge e in $\overline{G}$ if v is adjacent to at least one endpoint of the given edge; in particular, the two endpoints of e are close to e. Let $C_{\overline{G}}(e)$ be the set of vertices close to the edge e, and let $C_{\overline{G}}(v)$ be the set of edges that are close to the vertex v.

Observe that for a vertex v and an edge $e = \{x, y\}$ in $\overline{G}$, v is close to e if and only if no shortest path from x to y in G passes through v, and thus x and y do not contribute to the betweenness of v in G. In particular, for an edge $e = \{x, y\}$ of $\overline{G}$, there are $n - |C_{\overline{G}}(e)|$ shortest paths from x to y in G, each passing through a different vertex of $V \setminus C_{\overline{G}}(e)$. We denote the contribution of the edge e to each vertex of $V(G) \setminus C_{\overline{G}}(e)$ as the *weight* of the edge e, $w(e) = 1/(n - |C_{\overline{G}}(e)|)$.

The *weight* of a vertex v, $w(v) = \sum_{e \in C_{\overline{G}}(v)} w(e)$, is closely related to the betweenness of v.

Proposition 5. *Let G be of diameter two. Then for all $x \in V(G)$,*

$$B(x) = \Big(\sum_{e \in E} w(e)\Big) - w(x).$$

Corollary 6. *G is a BUG if and only if all vertices of $\overline{G}$ have the same weight.*

Furthermore, as we are interested only in betweenness values below one, $\overline{G}$ must have more vertices than edges by Lemma 2.

Observation 7. *If G is a BUG with $B(G) < 1$, then $\overline{G}$ has some components which are trees.*

Considering x a leaf adjacent to y, $x, y \in V(G)$, by comparing $C_{\overline{G}}(x)$ and $C_{\overline{G}}(y)$, we obtain a restriction on the structure of the tree components of $\overline{G}$.

Lemma 8. *For G a BUG, any vertex of degree one appears only in a star component of $\overline{G}$.*

Conditioning on the vertices having the same weight, we even prohibit stars of different sizes.

Proposition 9. *All the tree components of a coBUG $\overline{G}$ are stars of the same size.*

Let us call a BUG G *exotic* if it has $B(G) < 1$ and $\overline{G}$ contains a component different from a star.

Conjecture 10. *There are no exotic betweenness-uniform graphs.*

A graph $\overline{G}$ is *(m,t)-uniform* if $|C_{\overline{G}}(v)| = m$ for every $v \in V(\overline{G})$ and $|C_{\overline{G}}(e)| = t$ for every $e \in E(\overline{G})$. Note that if $\overline{G}$ is disconnected and (m, t)-uniform, then G is always a BUG with betweenness value m/t. Indeed, in an (m, t)-uniform graph $\overline{G}$, every vertex has weight $\frac{m}{n-t}$. However, we can show that there can be no exotic BUGs whose complements have nontrivial (m, t)-uniform components.

Theorem 11. *There is no exotic BUG with a complement containing an (m, t)-uniform component other than a star.*

Note that there exist infinitely many BUGs of betweenness exactly one whose complements have both a star component and an (m, t)-uniform non-star component. There are also BUGs whose complements have a star component and a non-(m, t)-uniform component, but we have not found any such BUG with betweenness below one.

3.3 Only values $\frac{k}{k+1}$ on the interval $\langle 0, \frac{6}{7} \rangle$

Apart from showing that a non-star component of a coBUG with density less than one would have to be non-(m,t)-uniform, we prove that some small stars cannot occur as a component of a coBUG with any other types of components, by which we show that there are no exotic BUGs with betweenness below $\frac{6}{7}$.

By comparing the vertices and edges in the closeness relation and their weights, we can infer the following forbidden structures in components of coBUGs.

Lemma 12. *Let H be a component of a coBUG. Then*

- *H has no vertex of degree one unless $H \approx K_{1,\ell}$ for some $\ell \geq 1$.*
- *H has no vertex of degree two whose neighbours are adjacent unless $H \approx K_{1,1,\ell}$ for some $\ell \geq 1$.*
- *H has no two adjacent vertices a, b of degree two, whose neighbours c and d are adjacent, i.e. $ab, ac, bd, cd \in E(H)$, unless $H \approx C_4$.*
- *H has no adjacent vertices a, b of degree two, whose neighbours e and c have a common neighbour d, i.e. $ab, bc, cd, de, ea \in H$, unless $H \approx C_5$*
- *H has no vertices a, b, c of degree two such that $ab, bc \in E(H)$*

By using Lemma 12 and by counting of edges we can conclude the following lemma.

Lemma 13. *Let v be a vertex of a non-star component of a coBUG $\overline{G}$. If* $\deg(v) = d$, *then $|C_{\overline{G}}(v)| \geq 2d$.*

By considering the weight $\frac{\ell}{n-(\ell+1)}$ of vertices in a star $K_{1,\ell}$, by careful consideration of obtainable weights, by using of Lemmas 12 and 13 and by a case analysis we can prove the following theorem.

Theorem 14. *Let H be one of the following: K_1, K_2, $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, $K_{1,5}$, $K_{1,6}$. If $\overline{G}$ is a coBUG of density less than 1 containing H as a connected component, then all the other components of $\overline{G}$ are isomorphic to H as well.*

This result allows us to both disprove the Conjecture 1 and to generalize previous result by Hurajová and Madaras [6] claiming that there are no betweenness-uniform graphs in the interval $(0, \frac{1}{2})$.

Corollary 15. *If G is a BUG with $B(G) \leq \frac{6}{7}$, then $B(G) \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{6}{7}\}$. Moreover, any such BUG is a complement of a disjoint union of stars of the same size.*

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References

[1] J. C. Birk, C. J. Bresloff, and R. Darbali-Zamora. Electric Grid Vulnerability Analysis to Identify Communities Prone to Wildfires. *IEEE Access*, 2023.

[2] C. C. Chou, C.-S. Teng, and H. H. Tung. How do alliances trade arms? Political alliance networks and global arms transfers. *PLOS ONE*, 18(3):1–14, 2023.

[3] F. Comellas and S. Gago. Spectral bounds for the betweenness of a graph. *Linear Algebra and its Applications*, 423(1):74 – 80, 2007. Special Issue devoted to papers presented at the Aveiro Workshop on Graph Spectra.

[4] L. Freeman. A set of measures of centrality based on betweenness. *Sociometry*, 40:35–41, 1977.

[5] S. Gago. *Métodos espectrales y nuevas medidas, modelos y parámetros en grafos pequeño-mundo invariantes de escala.* PhD thesis, Universitat Politècnica de Catalunya, 2006.

[6] S. Gago, J. Hurajova-Coronicova, and T. Madaras. On betweenness-uniform graphs. *Czechoslovak Mathematical Journal*, 63(3):629–642, 2013.

[7] D. Hartman, A. Pokorná, and P. Valtr. On the connectivity and the diameter of betweenness-uniform graphs. In A. Mudgal and C. R. Subramanian, editors, *Algorithms and Discrete Applied Mathematics*, pages 317–330, Cham, 2021. Springer.

[8] J. Hurajova-Coronicova and T. Madaras. More on betweenness-uniform graphs. *Czechoslovak Mathematical Journal*, 68(2):293–306, 2018.

[9] R. Lopez, J. Worrell, H. Wickus, R. Florez, and D. A. Narayan. Towards a characterization of graphs with distinct betweenness centralities. *Australasian Journal Of Combinatorics*, 68(2):285–303, 2017.

[10] A. Ouellet and M. Badri. Combining object-oriented metrics and centrality measures to predict faults in object-oriented software: An empirical validation. *Journal of Software: Evolution and Process*, 2023.

[11] A. Pokorná. Characteristics of network centralities. Master's thesis, Charles University, Prague, Czech Republic, 2020.

[12] Y. Sarabi and M. Smith. Gender diversity and publication activity—an analysis of STEM in the UK. *Research Evaluation*, 2023.