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Extremal number of graphs from geometric shapes

(Extended abstract)

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Abstract

We study the Turán problem for highly symmetric bipartite graphs arising from geometric shapes and periodic tilings commonly found in nature.

1. The prism $C_{2\ell}^{\square} := C_{2\ell} \square K_2$ is the graph consisting of two vertex disjoint 2ℓ-cycles and a matching pairing the corresponding vertices of these two cycles. We show that for every $\ell \geqslant 4$, $\mathrm{ex}(n, C_{2\ell}^{\square}) = \Theta(n^{3/2})$. This resolves a conjecture of He, Li and Feng.
2. The hexagonal tiling in honeycomb is one of the most natural structures in the real world. We show that the extremal number of honeycomb graphs has the same order of magnitude as their basic building unit 6-cycles.
3. We also consider bipartite graphs from quadrangulations of the cylinder and the torus. We prove near optimal bounds for both configurations. In particular, our method gives a very short proof of a tight upper bound for the extremal number of the 2-dimensional grid, improving a recent result of Bradač, Janzer, Sudakov and Tomon.

Our proofs mix several ideas, including shifting embedding schemes, weighted homomorphism and subgraph counts and asymmetric dependent random choice.

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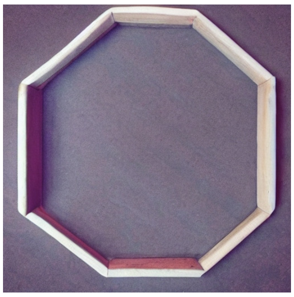
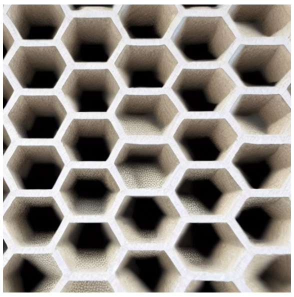
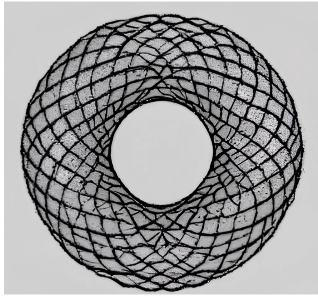

1 Introduction

The Turán problem, one of the most central topics in extremal combinatorics, is concerned with determining the maximum density of graphs without containing a given graph as a subgraph. Formally, for a graph F, the *extremal number of* F, denoted by $\mathrm{ex}(n, F)$, is the maximum number of edges in an n-vertex graph not containing F as a subgraph. The celebrated Erdős-Stone-Simonovits Theorem [9, 12] asymptotically solves the problem when $\chi(F) \geqslant 3$. However, for bipartite graphs, not even the order of magnitude is known in general. Turán [21] in 1941 proposed the study of the five graphs from platonic solids, and his result covers the tetrahedron graph K_4. The problem of octahedron, dodecahedron and icosahedron graphs were later resolved by Erdős and Simonovits [11] and by Simonovits [19, 20] respectively; while the innocent looking cube graph remains elusive. Two basic classes of bipartite graphs with high symmetry are even cycles and complete bipartite graphs; both of them have been widely studied for several decades [2, 3, 5, 6, 8, 15, 17, 22]. For more on the bipartite Turán problem, we refer the reader to the comprehensive survey of Füredi and Simonovits [13].

In this paper, we continue this line of study and determine the order of magnitude of the extremal number for several highly symmetric bipartite graphs stemming from certain geometric shapes and periodic tilings, including the prism, the grid, the honeycomb and certain quadrangulations of the cylinder and the torus.

1.1 The prisms

The 2ℓ-*prism* $C_{2\ell}^{\square} := C_{2\ell} \square K_2$ is the Cartesian product of 2ℓ-cycle with an edge, consisting of two vertex disjoint $C_{2\ell}$ and a matching joining the corresponding vertices on these two cycles. As $C_{2\ell}^{\square}$ contains many 4-cycles, we have a lower bound $\mathrm{ex}(n, C_{2\ell}^{\square}) \geqslant \mathrm{ex}(n, C_4) = \Omega(n^{3/2})$. Note that $C_4^{\square}$ is the notorious cube graph, for which the best known bounds are $\Omega(n^{3/2}) \leqslant \mathrm{ex}(n, C_4^{\square}) \leqslant O(n^{8/5})$ [10, 18]. Studying the 2ℓ-prism $C_{2\ell}^{\square}$ could shed some light on the cube problem. An upper bound $\mathrm{ex}(n, C_{2\ell}^{\square}) = O(n^{5/3})$ can be easily obtained via the celebrated dependent random choice method [1].

Very recently, He, Li and Feng [14] studied the odd prisms, determined $\mathrm{ex}(n, C_{2k+1}^{\square})$ for any $k \geqslant 1$ for large n and characterized the extremal graphs. They proposed the following conjecture to break the 5/3 barrier for 2ℓ-prism.

Conjecture 1 ([14]). *For every $\ell \geqslant 2$, there exists $c = c(\ell) > 0$ such that* $\mathrm{ex}(n, C_{2\ell}^{\square}) = O(n^{5/3-c})$.

Our first result provides an optimal upper bound for $C_{2\ell}^{\square}$ for every $\ell \geqslant 4$.

Theorem 1.1. *For any integer $\ell \geqslant 4$, we have*

$$\mathrm{ex}(n, C_{2\ell}^{\square}) = \Theta_\ell(n^{3/2}).$$

We remark that larger prisms are easier to handle. We can provide a shorter and different proof of $\mathrm{ex}(n, C_{2\ell}^{\square}) = O_\ell(n^{3/2})$ for $\ell \geqslant 7$, which can also be used to show that $\mathrm{ex}(n, C_6^{\square}) = O(n^{21/13}(\log n)^{24/13})$. This, together with the known bound for the cube and Theorem 1.1, proves Conjecture 1.

It is worth mentioning a closely related conjecture of Erdős. A graph is *r-degenerate* if each of its subgraphs has minimum degree at most r. Erdős [7] conjectured that for a bipartite H, $\mathrm{ex}(n, H) = O(n^{3/2})$ if and only if H is 2-degenerate. This conjecture was recently disproved by Janzer [16], who constructed, for each $\varepsilon > 0$, a 3-regular bipartite graph H with girth 6 such that $\mathrm{ex}(n, H) = O(n^{4/3+\varepsilon})$. Theorem 1.1 provides a family of 3-regular *girth-4* counterexamples.

1.2 The honeycomb

The hexagonal tiling in honeycomb is one of the most common geometric structures, appearing in nature in many crystals. It is also the densest way to pack circles in the plane. As the honeycomb graph H of any size contains C_6 as a subgraph, we have a lower bound $\mathrm{ex}(n, H) \geqslant \mathrm{ex}(n, C_6) = \Omega(n^{4/3})$.

Our second result is a matching upper bound $O(n^{4/3})$, showing that the hexagonal tiling appears soon after the appearance of a single hexagon. In particular, we consider the following graph $H_{k,\ell}$ (see Figure 1), which contains any (finite truncation of a) honeycomb graph as a subgraph when k and ℓ are sufficiently large.

Definition. For an odd integer $k \geqslant 1$ and even integer $\ell \geqslant 2$, let $H_{k,\ell}$ be the graph with vertex set $V(H_{k,\ell}) = \{x_{i,j} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell\}$, where $x_{k,1} = x_{k,3} = \cdots = x_{k,\ell-1} = u$ and $x_{1,2} = x_{1,4} = \cdots = x_{1,\ell} = v$ (but all the other vertices are distinct) and edge set

$$E(H_{k,\ell}) = \{x_{i,j}x_{i,j+1} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell - 1\} \cup \{x_{2i-1,j}x_{2i,j} : 1 \leqslant i \leqslant k/2, 1 \leqslant j \leqslant \ell,\ j \text{ is odd}\} \cup \{x_{2i,j}x_{2i+1,j} : 1 \leqslant i \leqslant k/2, 1 \leqslant j \leqslant \ell,\ j \text{ is even}\}.$$

Theorem 1.2. *For positive odd integers $k \geqslant 1$ and $\ell \geqslant 2$,*

$$\mathrm{ex}(n, H_{k,\ell}) = \Theta_{k,\ell}(n^{4/3}).$$

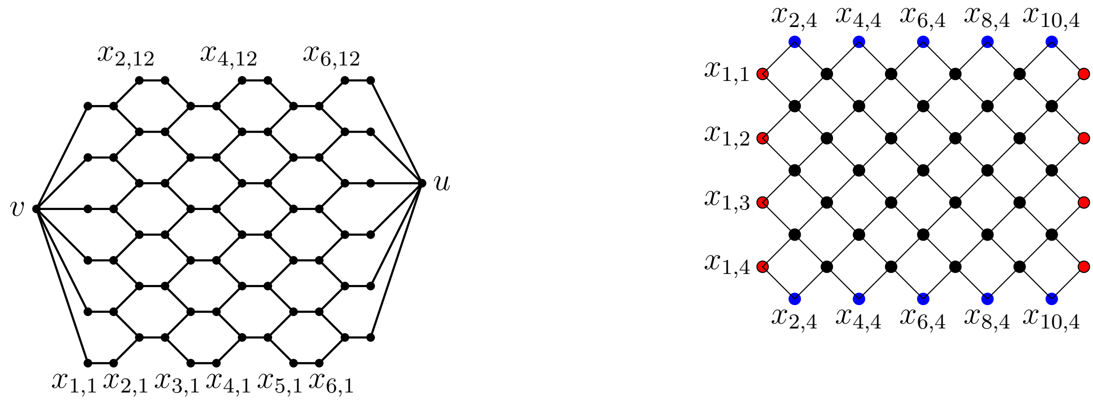

Figure 1: The first graph is $H_{7,12}$. In the second graph, identifying the blue vertices (in the same column) yields a copy of $P_{11,4}$; if additionally the red vertices (in the same row) are identified, then we obtain a copy of $T_{10,4}$.

1.3 The grid

We will also give an improved bound for the extremal number of the grid. For a positive integer t, $F_{t,t}$ is the graph with vertex set $[t] \times [t]$ in which two vertices are joined by an edge if they differ in exactly one coordinate and in that coordinate they differ by one. Bradač, Janzer, Sudakov and Tomon [4] determined the extremal number of $F_{t,t}$ up to a multiplicative constant which depends on t, showing that for any $t \geqslant 2$,

$$\Omega(t^{1/2}n^{3/2}) \leqslant \mathrm{ex}(n, F_{t,t}) \leqslant e^{O(t^5)}n^{3/2}.$$

They have asked to determine the correct dependence on t. We make substantial progress on this question by giving a very short proof of the following bound, which shows that the dependence on t is polynomial.

Theorem 1.3. *For any positive integer t, if n is sufficiently large in terms of t, then*

$$\mathrm{ex}(n, F_{t,t}) \leqslant 5t^{3/2}n^{3/2}.$$

It would be interesting to determine the correct power of t in $\mathrm{ex}(n, F_{t,t})$.

1.4 Quadrangulations of cylinder and torus

Next, we consider certain quadrangulations of the cylinder and the torus.

Definition (Quadrangulation of a cylinder)**.** For integers $k, \ell \geqslant 2$, let $P_{k,\ell}$ be the graph with vertex set $V(P_{k,\ell}) = \{x_{i,j} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell\}$, and edge set

$$E(P_{k,\ell}) = \{x_{i,j}x_{i+1,j} : 1 \leqslant i \leqslant k-1, 1 \leqslant j \leqslant \ell\} \cup \{x_{i,j+1}x_{i+1,j} : 1 \leqslant i \leqslant k-1, 1 \leqslant j \leqslant \ell,\ i \text{ is odd}\}$$
$$\cup \{x_{i,j}x_{i+1,j+1} : 1 \leqslant i \leqslant k-1, 1 \leqslant j \leqslant \ell,\ i \text{ is even}\},$$

where $x_{i,\ell+1} = x_{i,1}$ for all $i \in [k]$.

Clearly, the extremal number of such a quadrangulated cylinder is at least that of the 4-cycle. Our next result infers that in fact they are of the same order of magnitude.

Theorem 1.4. *Let k and ℓ be positive integers. Then we have*

$$\mathrm{ex}(n, P_{k,\ell}) = \Theta_{k,\ell}(n^{3/2}).$$

If k is even and we glue the two sides of the cylinder $P_{k+1,\ell}$, then we obtain a torus, see Figure 1.

Definition (Quadrangulation of a torus)**.** For an even integer $k \geqslant 4$ and integer $\ell \geqslant 2$, let $T_{k,\ell}$ be the graph with vertex set $V(T_{k,\ell}) = \{x_{i,j} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell\}$, and edge set

$$\begin{aligned} E(T_{k,\ell}) = \{x_{i,j}x_{i+1,j} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell\} &\cup \{x_{i,j+1}x_{i+1,j} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell,\ i \text{ is odd}\} \\ &\cup \{x_{i,j}x_{i+1,j+1} : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell,\ i \text{ is even}\}, \end{aligned}$$

where $x_{k+1,j} = x_{1,j}$ for all $j \in [\ell]$ and $x_{i,\ell+1} = x_{i,1}$ for all $i \in [k]$.

For the quadrangulated torus, we provide a general upper bound as follows.

Theorem 1.5. *For an even integer $k \geqslant 4$ and an integer $\ell \geqslant 2$, we have*

$$\mathrm{ex}(n, T_{k,\ell}) = O_{k,\ell}(n^{\frac{3}{2}+\frac{\ell}{k}}(\log n)^2).$$

Thus, when k is sufficiently large compared to ℓ, the exponent can be arbitrarily close to 3/2. On the other hand, the exponent is always strictly greater than 3/2 as the probabilistic deletion method yields the lower bound $\mathrm{ex}(n, T_{k,\ell}) = \Omega_{k,\ell}(n^{\frac{3}{2}+\frac{3}{4k\ell-2}})$.

2 Ideas of proofs

In this section, we briefly discuss some key ideas in our proofs.

2.1 Shifting embedding schemes: Grid, quadrangulated cylinder, torus and honeycomb

For Theorems 1.2, 1.3 and 1.4, our embedding strategy is based on the observation that, if we can find a large collection of paths or cycles with a certain nice property, then we can repeatedly replace the vertices (or edges) of the chosen paths or cycles with vertices (or edges) from a new one in the collection to build the desired tilings. Formally, the definition of an α-rich collection of paths in a graph is as follows.

Definition 2.1. Let $\alpha > 0$ and $k \in \mathbb{N}$. We say that a collection $\mathcal{P}$ of (labelled) paths P_k is *α-rich* if for any member $x_1x_2\cdots x_k \in \mathcal{P}$ and any $2 \leqslant i \leqslant k-1$, there exist at least α distinct vertices x_i' such that $x_1x_2\cdots x_{i-1}x_i'x_{i+1}\cdots x_k \in \mathcal{P}$.

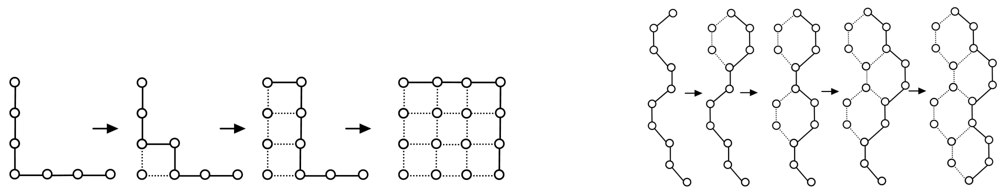

Figure 2: The process to build grids and honeycomb graphs.

Finding a $t \times t$ grid then boils down to constructing an α-rich collection of paths of length $2t-2$ with sufficiently large α; see the left side of Figure 2 for an illustration. In order to find the desired quadrangulations of the cylinder and the torus, we use an analogous definition of rich *cycles*.

To deal with honeycomb graphs, we introduce the following definition.

Definition 2.2. Let $\alpha > 0$ and $k \in \mathbb{N}$. A collection $\mathcal{P}$ of paths P_k is *α-good* if the following holds. For any $x_1x_2 \cdots x_k \in \mathcal{P}$ and $2 \leqslant i \leqslant k-2$, there are at least α pairwise disjoint edges $x_i'x_{i+1}'$ such that $x_1x_2 \cdots x_{i-1}x_i'x_{i+1}'x_{i+2} \cdots x_k \in \mathcal{P}$.

Finding a honeycomb graph boils down to constructing an α-good collection of paths for a sufficiently large α; see the right side of Figure 2 for an illustration.

While it is not too hard to find a collection of rich paths and even cycles via supersaturation, it is a lot more challenging to construct a collection of good paths. In order to accomplish the latter, rather than doing a direct counting using supersaturation, we carry out a weighted count.

2.2 Weighted count of homomorphisms for Theorem 1.1

In this subsection, we give a brief outline of the proof of Theorem 1.1. Let us call an n-vertex graph H with average degree d *clean* if for any $uv \in E(H)$, u has at least $d/16$ neighbours w in H such that $d_H(v,w) \geqslant \frac{d^2}{128n}$. It can be shown that any graph with average degree at least $2d$ contains a clean subgraph with average degree at least d.

Let G be a graph of average degree d and let distinct vertices x_i, y_i for $0 \leqslant i \leqslant \ell$ form a copy of $P_{\ell+1}^{\square} := K_2 \square P_{\ell+1}$, where $x_iy_i \in E(G)$ for every i and $x_{i-1}x_i, y_{i-1}y_i \in E(G)$ for every $1 \leqslant i \leqslant \ell$. Now the *weight* of this copy is defined to be $1/\prod_{i=1}^{\ell} \max(d_G(x_{i-1}, y_i), \frac{d^2}{n})$. For distinct vertices u, v, w, z, we call the 4-tuple (u, v, w, z) *rich* if $uv, wz \in E(G)$, and moreover there are at least 4ℓ pairwise vertex-disjoint edges $xy \in E(G)$ such that $ux, xw, vy, yz \in E(G)$. We say that vertices x_i, y_i (for $0 \leqslant i \leqslant \ell$) form a *nice* copy of $P_{\ell+1}^{\square}$ if they form a copy of $P_{\ell+1}^{\square}$, for every $1 \leqslant i \leqslant \ell$ the codegrees satisfy $d(x_{i-1}, y_i), d(x_i, y_{i-1}) \leqslant C_0 d^{1/2}$ (for some suitably defined constant C_0), and for every $2 \leqslant i \leqslant \ell$, the 4-tuple $(x_{i-2}, y_{i-2}, x_i, y_i)$ is not rich. We also say that vertices x_i, y_i, x_i', y_i' (for $0 \leqslant i \leqslant \ell$) form a *nice* homomorphic copy of $C_{2\ell}^{\square}$ if $x_0 = x_0', y_0 = y_0', x_\ell = x_\ell', y_\ell = y_\ell'$, both $\{x_i, y_i : 0 \leqslant i \leqslant \ell\}$ and $\{x_i', y_i' : 0 \leqslant i \leqslant \ell\}$ form a nice copy of $P_{\ell+1}^{\square}$, each x_i is distinct from all other vertices except possibly x_i' and each y_i is distinct from all other vertices except possibly y_i'. We define the weight of a homomorphic copy of $C_{2\ell}^{\square}$ to be $\left(\prod_{i=1}^{\ell}\max(d(x_{i-1},y_i),\frac{d^2}{n})\cdot\prod_{i=1}^{\ell}\max(d(x_{i-1}',y_i'),\frac{d^2}{n})\right)^{-1}$.

Let G be a clean, bipartite, n-vertex graph with average degree $d \geqslant Cn^{1/2}$ and maximum degree at most Kd, where K is some absolute constant and C is a sufficiently large constant (which can depend on ℓ). Our proof consists of the following steps.

1. We first prove that the total weight of nice copies of $P_{\ell+1}^{\square}$ in G is at least $\Omega_\ell(nd^{\ell+1})$.
2. Noting that by gluing together two nice copies of $P_{\ell+1}^{\square}$, we get a nice homomorphic copy of $C_{2\ell}^{\square}$, one can easily deduce from step 1 that the total weight of nice homomorphic copies of $C_{2\ell}^{\square}$ in G is $\Omega_\ell(d^{2\ell})$.
3. By carefully analyzing different types of degenerate homomorphic copies of $C_{2\ell}^{\square}$, we can show that for $\ell \geqslant 4$, the total weight of degenerate nice homomorphic copies of $C_{2\ell}^{\square}$ in G is at most $O(nd^{2\ell-2})$. This is negligible compared to $\Omega_\ell(d^{2\ell})$, showing that G contains a genuine copy of $C_{2\ell}^{\square}$.

3 Open problem

An open problem left in this paper is determining the extremal number of $C_6^{\square}$. We conjecture that $\mathrm{ex}(n, C_6^{\square}) = \Theta(n^{3/2})$.

References

[1] Noga Alon, Michael Krivelevich, and Benny Sudakov. Turán numbers of bipartite graphs and related Ramsey-type questions. *Combinatorics, Probability and Computing*, 12(5-6):477–494, 2003.

[2] Noga Alon, Lajos Rónyai, and Tibor Szabó. Norm-graphs: variations and applications. *J. Combin. Theory Ser. B*, 76(2):280–290, 1999.

[3] J. A. Bondy and M. Simonovits. Cycles of even length in graphs. *J. Combin. Theory Ser. B*, 16:97–105, 1974.

[4] Domagoj Bradač, Oliver Janzer, Benny Sudakov, and István Tomon. The Turán number of the grid. *Bull. Lond. Math. Soc.*, 55:194–204, 2023.

[5] W. G. Brown. On graphs that do not contain a Thomsen graph. *Canadian Mathematical Bulletin*, 9(3):281–285, 1966.

[6] Boris Bukh. Extremal graphs without exponentially-small bicliques. *arXiv preprint*, arXiv: 2109.04167, 2021.

[7] P. Erdős. Problems and results in graph theory. In *The theory and applications of graphs (Kalamazoo, Mich., 1980)*, pages 331–341. Wiley, New York, 1981.

[8] P. Erdős, A. Rényi, and V. T. Sós. On a problem of graph theory. *Studia Sci. Math. Hungar.*, 1:215–235, 1966.

[9] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar.*, 1:51–57, 1966.

[10] P. Erdős and M. Simonovits. Some extremal problems in graph theory. In *Combinatorial theory and its applications, I (Proc. Colloq., Balatonfüred, 1969)*, pages 377–390. North-Holland, Amsterdam, 1970.

[11] P. Erdős and M. Simonovits. An extremal graph problem. *Acta Math. Acad. Sci. Hungar.*, 22:275–282, 1971/72.

[12] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[13] Zoltán Füredi and Miklós Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdős centennial*, volume 25 of *Bolyai Soc. Math. Stud.*, pages 169–264. János Bolyai Math. Soc., Budapest, 2013.

[14] Xiaocong He, Yongtao Li, and Lihua Feng. Extremal graphs for the odd prism. *arXiv preprint*, arXiv: 2302.03278, 2023.

[15] Zhiyang He. A new upper bound on the Turán number of even cycles. *Electron. J. Combin.*, 28(2):Paper No. 2.41, 18, 2021.

[16] Oliver Janzer. Disproof of a conjecture of Erdős and Simonovits on the Turán number of graphs with minimum degree 3. *Int. Math. Res. Not. IMRN*, 2022, to appear.

[17] János Kollár, Lajos Rónyai, and Tibor Szabó. Norm-graphs and bipartite Turán numbers. *Combinatorica*, 16(3):399–406, 1996.

[18] Rom Pinchasi and Micha Sharir. On graphs that do not contain the cube and related problems. *Combinatorica*, 25(5):615–623, 2005.

[19] M. Simonovits. Extermal graph problems with symmetrical extremal graphs. Additional chromatic conditions. *Discrete Math.*, 7:349–376, 1974.

[20] Miklós Simonovits. The extremal graph problem of the icosahedron. *J. Combinatorial Theory Ser. B*, 17:69–79, 1974.

[21] P. Turán. Eine extremalaufgabe aus der graphentheorie. *Fiz Lapok*, pages 436–452, 1941.

[22] R. Wenger. Extremal graphs with no C^4's, C^6's, or C^{10}'s. *J. Combin. Theory Ser. B*, 52(1):113–116, 1991.