A GENERAL BOUND FOR THE INDUCED POSET SATURATION PROBLEM

(EXTENDED ABSTRACT)

Andrea Freschi[*]   Simón Piga[†]   Maryam Sharifzadeh[‡]
Andrew Treglown[§]

Abstract

For a fixed poset P, a family $\mathcal{F}$ of subsets of $[n]$ is induced P-saturated if $\mathcal{F}$ does not contain an induced copy of P, but for every subset S of $[n]$ such that $S \notin \mathcal{F}$, then P is an induced subposet of $\mathcal{F} \cup \{S\}$. The size of the smallest such family $\mathcal{F}$ is denoted by $\mathrm{sat}^*(n, P)$. Keszegh, Lemons, Martin, Pálvölgyi and Patkós [Journal of Combinatorial Theory Series A, 2021] proved that there is a dichotomy of behaviour for this parameter: given any poset P, either $\mathrm{sat}^*(n, P) = O(1)$ or $\mathrm{sat}^*(n, P) \geq \log_2 n$. We improve this general result showing that either $\mathrm{sat}^*(n, P) = O(1)$ or $\mathrm{sat}^*(n, P) \geq 2\sqrt{n-2}$. Our proof makes use of a Turán-type result for digraphs.

Curiously, it remains open as to whether our result is essentially best possible or not. On the one hand, a conjecture of Ivan states that for the so-called diamond poset $\Diamond$ we have $\mathrm{sat}^*(n, \Diamond) = \Theta(\sqrt{n})$; so if true this conjecture implies our result is tight up to a multiplicative constant. On the other hand, a conjecture of Keszegh, Lemons, Martin, Pálvölgyi and Patkós states that given any poset P, either $\mathrm{sat}^*(n, P) = O(1)$ or $\mathrm{sat}^*(n, P) \geq n + 1$. We prove that this latter conjecture is true for a certain class of posets P.

[*]University of Birmingham, United Kingdom. E-mail: axf079@bham.ac.uk.

[†]University of Birmingham, United Kingdom. E-mail: s.piga@bham.ac.uk. Research supported by EPSRC grant EP/V002279/1.

[‡]Umeå Universitet, Sweden. E-mail: maryam.sharifzadeh@umu.se.

[§]University of Birmingham, United Kingdom. E-mail: a.c.treglown@bham.ac.uk. Research supported by EPSRC grant EP/V002279/1.

1 Introduction

Saturation problems have been well studied in graph theory. A graph G is H-*saturated* if it does not contain a copy of the graph H, but adding any edge to G from its complement creates a copy of H. Turán's celebrated theorem [15] can be stated in the language of saturation: it determines the maximum number of edges in a K_r-saturated n-vertex graph. In contrast, Erdős, Hajnal and Moon [5] determined the minimum number of edges in a K_r-saturated n-vertex graph; see the survey [3] for further results in this direction.

In recent years there has been an emphasis on developing the theory of saturation for posets. Turán-type problems have been extensively studied in this setting (see, e.g., the survey [9]). In this paper we are interested in *minimum* saturation questions à la Erdős–Hajnal–Moon. In particular, we consider *induced saturation* problems.

All posets we consider will be (implicitly) viewed as finite collections of finite subsets of $\mathbb{N}$. In particular, we say that P is a *poset on* $[p] := \{1, 2, \ldots, p\}$ if P consists of subsets of $[p]$. Let P, Q be posets. A *poset homomorphism* from P to Q is a function $\phi: P \to Q$ such that for every $A, B \in P$, if $A \subseteq B$ then $\phi(A) \subseteq \phi(B)$. We say that P is a *subposet* of Q if there is an injective poset homomorphism from P to Q; otherwise, Q is said to be P-*free*. Further we say P is an *induced subposet* of Q if there is an injective poset homomorphism ϕ from P to Q such that for every $A, B \in P$, $\phi(A) \subseteq \phi(B)$ if and only if $A \subseteq B$; otherwise, Q is said to be *induced* P-*free*.

For a fixed poset P, we say that a family $\mathcal{F} \subseteq 2^{[n]}$ of subsets of $[n]$ is P-*saturated* if $\mathcal{F}$ is P-free, but for every subset S of $[n]$ such that $S \notin \mathcal{F}$, then P is a subposet of $\mathcal{F} \cup \{S\}$. A family $\mathcal{F} \subseteq 2^{[n]}$ of subsets of $[n]$ is *induced* P-*saturated* if $\mathcal{F}$ is induced P-free, but for every subset S of $[n]$ such that $S \notin \mathcal{F}$, then P is an induced subposet of $\mathcal{F} \cup \{S\}$.

The study of minimum saturated posets was initiated by Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi and Patkós [8] in 2013. In their work the relevant parameter is $\mathrm{sat}(n, P)$, which is defined to be the size of the smallest P-saturated family of subsets of $[n]$. See, e.g., [8, 12, 14] for various results on $\mathrm{sat}(n, P)$.

The induced analogue of $\mathrm{sat}(n, P)$ – denoted by $\mathrm{sat}^*(n, P)$ – was first considered by Ferrara, Kay, Kramer, Martin, Reiniger, Smith and Sullivan [6]. Thus, $\mathrm{sat}^*(n, P)$ is defined to be the size of the smallest induced P-saturated family of subsets of $[n]$. The following result from [12] (and implicit in [6]) shows that the parameter $\mathrm{sat}^*(n, P)$ has a dichotomy of behaviour.

Theorem 1.1. *[6, 12] For any poset P, either there exists a constant K_P with $\mathrm{sat}^*(n, P) \leq K_P$ or $\mathrm{sat}^*(n, P) \geq \log_2 n$, for all $n \in \mathbb{N}$.*

Probably the most important open problem in the area is to obtain a tight version of Theorem 1.1; that is, to replace the $\log_2 n$ in Theorem 1.1 with a term that is as large as possible. In fact, Keszegh, Lemons, Martin, Pálvölgyi and Patkós [12] made the following conjecture in this direction.

Conjecture 1.2. *[12] For any poset P, either there exists a constant K_P with $\mathrm{sat}^*(n, P) \leq K_P$ or $\mathrm{sat}^*(n, P) \geq n + 1$, for all $n \in \mathbb{N}$.*

Note that the lower bound of $n + 1$ is rather natural here. For example, it is the size of the largest *chain* in $2^{[n]}$ as well as the smallest possible size of the union of two consecutive 'layers' in $2^{[n]}$, namely the layer containing $[n]$ and the layer containing all subsets of $[n]$ of size exactly $n - 1$. Furthermore, such structures form minimum induced saturated families for the so-called fork poset $\vee$, i.e., $sat^*(n, \vee) = n + 1$ [6]; so the lower bound in Conjecture 1.2 cannot be increased. There are also no known examples of posets P for which $sat^*(n, P) = \omega(n)$.

In contrast, Ivan [11, Section 3] presented evidence that led her to conjecture a rather different picture for the *diamond* poset $\Diamond$ (see Figure 1 for the Hasse diagram of $\Diamond$).

Conjecture 1.3. *[11] $sat^*(n, \Diamond) = \Theta(\sqrt{n})$.*

Our main result is the following improvement of Theorem 1.1.

Theorem 1.4. *For any poset P, either there exists a constant K_P with $sat^*(n, P) \leq K_P$ or $sat^*(n, \mathcal{P}) \geq 2\sqrt{n - 2}$, for all $n \in \mathbb{N}$.*

Thus, if Conjecture 1.3 is true, the lower bound in Theorem 1.4 would be tight up to a multiplicative constant.
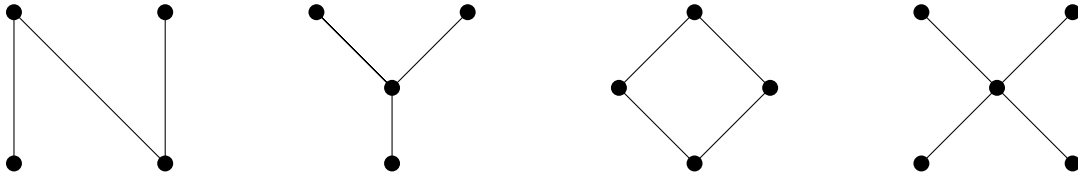

Figure 1: Hasse diagrams for the posets N, Y, $\Diamond$ and X.

On the other hand, we prove that Conjecture 1.2 does hold for a class of posets (that does not include $\Diamond$). Given $p \in \mathbb{N}$ and a poset P on $[p]$ we define the *dual* $\overline{P}$ of P as $\overline{P} := \{[p] \setminus F \colon F \in P\}$. We say a poset P *has legs* if there are distinct elements $L_1, L_2, H \in P$ such that L_1, L_2 are incomparable, $L_1, L_2 \subseteq H$ and for any other element $A \in P \setminus \{L_1, L_2, H\}$ we have $A \supseteq H$. The elements L_1 and L_2 are called *legs* and H is called a *hip*.

Theorem 1.5. *Let P be a poset with legs and $n \geq 3$. Then $sat^*(n, P) \geq n + 1$. Moreover, if both P and $\overline{P}$ have legs, then $sat^*(n, P) \geq 2n + 2$.*

Our results still leave both Conjecture 1.2 and Conjecture 1.3 open, and it is unclear to us which of these conjectures is true. However, if Conjecture 1.3 is true we believe it highly likely that there will be other posets P for which $sat^*(n, P) = \Theta(\sqrt{n})$.

It is also natural to seek exact results on $sat^*(n, P)$. However, despite there already being several papers concerning $sat^*(n, P)$ [1, 4, 6, 10, 11, 12, 13], there are relatively few posets P for which $sat^*(n, P)$ is known precisely (see Table 1 in [12] for a summary of most of the known results). Our next result extends this limited pool of posets, determining $sat^*(n, X)$ and $sat^*(n, Y)$ (see Figure 1 for the Hasse diagrams of X and Y).

Theorem 1.6. *Given any $n \in \mathbb{N}$ with $n \geq 3$,*

   (i) $sat^*(n, Y) = n + 2$ *and*

  (ii) $sat^*(n, X) = 2n + 2$.

Note that Theorem 1.6(*ii*) easily follows via Theorem 1.5 and an extremal construction. An application of Theorem 1.5 to Y only yields that $sat^*(n, Y) \geq n + 1$, so we require an extra idea to obtain Theorem 1.6(*i*).

It is natural to consider induced saturation problems for families of posets. Given a family of posets $\mathcal{P}$, we say that $\mathcal{F} \subseteq 2^{[n]}$ is *induced $\mathcal{P}$-saturated* if $\mathcal{F}$ contains no induced copy of any poset $P \in \mathcal{P}$ and for every $S \in 2^{[n]} \setminus \mathcal{F}$ there exists an induced copy of some poset $P \in \mathcal{P}$ in $\mathcal{F} \cup \{S\}$. We denote the size of the smallest such family by $sat^*(n, \mathcal{P})$. By following the proof of Theorem 1.4 precisely, one obtains the following result.

Theorem 1.7. *For any family of posets $\mathcal{P}$, either there exists a constant $K_{\mathcal{P}}$ with $sat^*(n, \mathcal{P}) \leq K_{\mathcal{P}}$ or $sat^*(n, \mathcal{P}) \geq 2\sqrt{n-2}$, for all $n \in \mathbb{N}$.*

In light of Theorem 1.7 it is natural to ask whether an analogue of Conjecture 1.2 is true in this more general setting, or whether (for example) the lower bound on $sat^*(n, \mathcal{P})$ in Theorem 1.7 is best possible up to a multiplicative constant.

The proofs of Theorems 1.4–1.7 appear in [7]. In the next section we describe how we make use of a Turán-type result for digraphs in the proof of Theorem 1.4.

2   A connection to a Turán problem for digraphs

In [13] a trick was introduced which can be used to prove lower bounds on $sat^*(n, P)$ for some posets P. The idea is to construct a certain auxiliary digraph D whose vertex set consists of the elements in an induced P-saturated family $\mathcal{F}$; one then argues that how this digraph is defined forces D to contain many edges, which in turn forces a bound on the size of the vertex set of D (i.e., lower bounds $|\mathcal{F}|$). This trick has been used to prove that $sat^*(n, \Diamond) \geq \sqrt{n}$ [13, Theorem 6] and $sat^*(n, N) \geq \sqrt{n}$ [10, Proposition 4] (see Figure 1 for the Hasse diagram of N).

Our proof of Theorem 1.4 utilises a variant of this digraph trick. In particular, by introducing an appropriate modification to the auxiliary digraph D used in [13], we are able to deduce certain Turán-type properties of D. Turán problems in digraphs are classical in extremal combinatorics and their study can be traced back to the work of Brown and Harary [2]. In [7] we prove a Turán-type result concerning *transitive cycles*, stated as Theorem 2.1 below.

Given $k \geq 3$, the *transitive cycle on k vertices* $\overrightarrow{TC}_k$ is a digraph with vertex set $[k]$ and every directed edge from i to $i + 1$ for every $i \in [k-1]$, as well as the directed edge from 1 to k. We establish an upper bound on the number of edges of a digraph not containing any transitive cycle.

Theorem 2.1. *Let $n \in \mathbb{N}$ and let D be a digraph on n vertices. If D is $\overrightarrow{TC}_k$-free for all $k \geq 3$, then*

$$e(D) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 2.$$

Note that the bound in Theorem 2.1 is best possible up to an additive constant. Indeed, consider the n-vertex digraph D with vertex classes A, B of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ respectively and all possible directed edges from A to B. So D has $\lfloor n^2/4 \rfloor$ edges and contains no transitive cycle.

Data availability statement. A full paper containing the proofs of our results can be found on arXiv [7].

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