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Upper bounds on Ramsey numbers for vector spaces over finite fields

(Extended abstract)

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Abstract

For $B \subseteq \mathbb{F}_q^m$, let $\mathrm{ex}_{\mathrm{aff}}(n, B)$ denote the maximum cardinality of a set $A \subseteq \mathbb{F}_q^n$ with no subset which is affinely isomorphic to B. Furstenberg and Katznelson proved that for any $B \subseteq \mathbb{F}_q^m$, $\mathrm{ex}_{\mathrm{aff}}(n, B) = o(q^n)$ as $n \to \infty$. For certain q and B, some more precise bounds are known. We connect some of these problems to certain Ramsey-type problems, and obtain some new bounds for the latter. For $s, t \geq 1$, let $R_q(s, t)$ denote the minimum n such that in every red-blue coloring of one-dimensional subspaces of $\mathbb{F}_q^n$, there is either a red s-dimensional subspace of $\mathbb{F}_q^n$ or a blue t-dimensional subspace of $\mathbb{F}_q^n$. The existence of these numbers is implied by the celebrated theorem of Graham, Leeb, Rothschild. We improve the best known upper bounds on $R_2(2, t)$, $R_3(2, t)$, $R_2(t, t)$, and $R_3(t, t)$.

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1 Main Results

We consider bounds for Ramsey-type and Turán-type problems in the setting of vector-spaces over finite fields. In this paper, we use $\begin{bmatrix} V \\ t \end{bmatrix}$ to denote the collection of all t-dimensional linear subspaces of a vector space V. The following theorem is a special case of a classical theorem of Graham, Leeb, and Rothschild [12], which proves the existence of the Ramsey numbers we consider.

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Theorem 1 (Graham, Leeb, Rothschild)**.** *Let $\mathbb{F}_q$ be any finite field. For any positive integers $t_1, \ldots, t_k$, there exists a minimum $n =: R_q(t_1, \ldots, t_k)$ such that for every k-coloring $f : \left[\begin{smallmatrix}\mathbb{F}_q^n \\ 1\end{smallmatrix}\right] \to [k]$ of the 1-dimensional linear subspaces of $\mathbb{F}_q^n$, there exist $i \in [k]$ and a linear subspace $U \subseteq \mathbb{F}_q^n$ of dimension t_i, such that $\left[\begin{smallmatrix}U \\ 1\end{smallmatrix}\right]$ is monochromatic in color i.*

In the case $t_1 = \cdots = t_k = t$, we write $R_q(t_1, \ldots, t_k) = R_q(t; k)$. The bounds for $R_q(t_1, \ldots, t_k)$ implied by early proofs of Theorem 1 (see [12], [18]) are quite large due to repeated use of the Hales-Jewett Theorem [13]. In the case $q = 2$, the problem can be reduced to the disjoint unions problem for finite sets, considered by Taylor [19], which gives the following bound.

Theorem 2 (Taylor)**.** *The number $R_2(t; k)$ is at most a tower of height $2k(t-1)$ of the form*

$$R_2(t;k) \leq k^{3^{k^{\cdot^{\cdot^{\cdot^{3}}}}}}.$$

For comparison, lower bounds attained from applying the techniques from [1] such as the Lovász Local Lemma to a uniform random coloring are only on the order of

$$R_2(t;k) = \Omega\left(\frac{2^t}{t}\log_2 k\right).$$

We improve the bound of Theorem 2 by removing the 3's from the tower.

Theorem 3. *For any t, k, $R_2(t; k)$ is at most a tower of height $k(t-1)$ of the form*

$$R_2(t;k) \leq k^{k^{\cdot^{\cdot^{\cdot^{k}}}}}.$$

More recently, Nelson and Nomoto considered the off-diagonal version of this problem over $\mathbb{F}_2$ with two colors, and they proved the following bound.

Theorem 4 (Nelson, Nomoto)**.** *For every $t \geq 2$,*

$$R_2(2,t) \leq (t+1)2^t.$$

Similar probabilistic arguments to those mentioned after Theorem 2 only give lower bounds linear in t for $R_2(2,t)$. Nelson and Nomoto asked if a subexponential upper bound is possible. While the answer to that question remains to be seen, we provide the following exponential improvement.

Theorem 5. *There exists a constant C such that for all $t \geq 2$,*

$$R_2(2,t) \leq Ct6^{t/4}.$$

We obtain the following analogous results over $\mathbb{F}_3$, using the same methodology.

Theorem 6. *There exists an absolute polynomial $p(x)$ such that for any t, k, $R_3(t; k)$ is at most a tower of height $k(t-1)$ of the form*

$$R_3(t;k) \leq p(k)^{p(k)^{\cdot^{\cdot^{\cdot^{p(k)}}}}}.$$

Theorem 7. *There exist constants C and A, with $A \approx 13.901$ such that for all $t \geq 2$,*

$$R_3(2,t) \leq CtA^t.$$

2 Background and Methodology

Before we discuss the proofs of these results, we give a brief introduction to affine extremal numbers, which are our principal tool. We say that a subset $A \subseteq \mathbb{F}_q^n$ contains an *affine copy* of $B \subseteq \mathbb{F}_q^m$ if there is an injective affine map $f : \mathbb{F}_q^m \to \mathbb{F}_q^n$ with $f(B) \subseteq A$. If $\mathcal{B} = \{B_i\}_{i \in I}$ is a family of subsets $B_i \subseteq \mathbb{F}_q^{m_i}$, we say that A is $\mathcal{B}$*-free* if A has no affine copy of any B_i. The largest size $\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B})$ of a $\mathcal{B}$-free subset of $\mathbb{F}_q^n$ is called the nth *affine extremal number* of $\mathcal{B}$. If $\mathcal{B} = \{B\}$, we write $\mathrm{ex}_{\mathrm{aff}}(n, \{B\}) = \mathrm{ex}_{\mathrm{aff}}(n, B)$. Determining these affine extremal numbers dates back at least to the following theorem of Furstenberg and Katznelson [8].

Theorem 8 (Furstenberg, Katznelson)**.** *Let $\mathbb{F}_q$ be any finite field. For any positive integer t,*

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_q^t) = o(q^n).$$

Since any $\mathcal{B}$-free set is $\mathbb{F}_q^t$-free for some t, Theorem 8 says that affine extremal numbers are always $o(q^n)$. Furstenberg and Katznelson went on to prove a density version of the Hales-Jewett Theorem [9], from which Theorem 8 is immediate. Alternative proofs of these results can be found in [16] and [15], respectively.

The projective version of this problem is even older, beginning with the following result of Bose and Burton [3].

Theorem 9 (Bose, Burton)**.** *Let $\mathbb{F}_q$ be a finite field, and let $t \geq 1$. Let A be a subset of $\left[\begin{smallmatrix}\mathbb{F}_q^n\\1\end{smallmatrix}\right]$ for which there is no linear t-dimensional subspace $U \subseteq \mathbb{F}_q^n$ with $\left[\begin{smallmatrix}U\\1\end{smallmatrix}\right] \subseteq A$. Then*

$$|A| \leq \frac{q^n - q^{n-t+1}}{q-1},$$

with equality if and only if $\left[\begin{smallmatrix}\mathbb{F}_q^n\\1\end{smallmatrix}\right] \setminus A = \left[\begin{smallmatrix}W\\1\end{smallmatrix}\right]$ for some linear $(n-t+1)$-dimensional linear subspace $W \subseteq \mathbb{F}_q^n$.

The problem of determining projective extremal numbers asymptotically for general projective configurations over $\mathbb{F}_q$ was solved by Geelen and Nelson [10], who proved a theorem analogous to the Erdős-Stone-Simonivits Theorem for graphs.

Returning to the affine context, it is unknown in general (see [11], Open Problem 32) whether the $o(q^n)$ term in Theorem 8 can be taken to be of the form $(q^{1-\varepsilon})^n$ for some $\varepsilon = \varepsilon(q,t) > 0$. However, for $q = 2$ and $q = 3$, we have the following respective results of Bonin and Qin [2], and of Fox and Pham [7].

Theorem 10 (Bonin, Qin)**.** *There exists an absolute constant c such that for every $t \geq 1$, every subset of $\mathbb{F}_2^n$ of size at least $(2^{1-c2^{-t}})^n$ contains an affine t-space.*

Later on, we will use a more precise bound implied by their argument, namely

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_2^t) < 2^{(1-2^{1-t})n+1}.$$

Theorem 11 (Fox, Pham). *There exist absolute constants c and C_0, with $C_0 \approx 13.901$ such that for every $t \geq 1$, every subset of $\mathbb{F}_3^n$ of size at least $\left(3^{1-cC_0^{-t}}\right)^n$ contains an affine t-space.*

The proof of Theorem 10 is entirely self-contained and is no more than a page. Theorem 11, on the other hand, is the culmination of several breakthroughs related to the Cap Set Problem, starting with the advances in polynomial methods from Croot, Lev, and Pach [4] and the subsequent proof of the Cap Set Theorem by Ellenberg and Gijswijt [5], which says that $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_3^1) \leq (3^{1-\varepsilon})^n$ for some $\varepsilon > 0$. Fox and Lovász [6] then proved a supersaturation version of this result, from which Fox and Pham derived Theorem 11, which is a multidimensional extension of the Cap Set Theorem. It is unknown whether the constant C_0 given in the theorem is tight, as probabilistic lower bounds for $\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_3^t)$ are on the order of $\left(3^{1-3^{-(1+o(1))t}}\right)^n$ [7].

3 Proof Outlines

We now show that Theorems 3, 6, and 7 are easy consequences of the affine extremal results Theorem 10 and Theorem 11. To prove Theorem 5, we prove an additional extremal result over $\mathbb{F}_2$ by way of supersaturation and some observations about sumsets and products of affine structures.

To begin, we show how Theorem 3 follows from Bonin and Qin's result, Theorem 10.

Proof of Theorem 3. Since $R_2(1, \ldots, 1) = 1$, and we can reasonably define $R_2(t_1, \ldots, t_k) = 0$ if some $t_i = 0$, it suffices to show that

$$R_2(t_1, t_2, \ldots, t_k) \leq (\log_2 k) 2^r,$$

where $r = \max_{i \leq k} R_2(t_1, \ldots, t_i - 1, \ldots, t_k)$. In this case, we get by induction that

$$R_2(t_1, \ldots, t_k) \leq (\log_2 k) k^{k^{\cdot^{\cdot^{\cdot^{k^2}}}}} \leq k^{k^{\cdot^{\cdot^{\cdot^{k^k}}}}},$$

where the height of the tower is $\sum_{i \leq k}(t_i - 1)$. Let $n = (\log_2 k)2^r$, and consider a k-coloring of $\left[{\mathbb{F}_2^n \atop 1}\right]$, which we view as a k-coloring of $\mathbb{F}_2^n \setminus \{0\}$. Without loss of generality, assume that at least $2^n/k = 2^{n - \log_2 k}$ points are given color 1. By our choice of n and Theorem 10, we have

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_2^r) < 2^{(1-2^{1-r})n+1} = 2^{n - 2\log_2 k + 1} \leq 2^{n - \log_2 k},$$

so there is an affine r-dimensional subspace A which is monochromatic in color 1. Note that $0 \notin A$ since 0 was not given a color. Let W be the translate of A containing 0, which is a linear r-space. Suppose that there is no linear t_i-space U_i with $U_i \setminus \{0\}$ monochromatic in color i for any $i \geq 2$. Then by our choice of r, there exists a linear $(t_1 - 1)$-space $U_1' \subseteq W$ with $U_1' \setminus \{0\}$ monochromatic in color 1. Let $u \in A$, and take $U_1 = \mathrm{span}\{U_1', u\}$, which is a linear t_1-space contained in $U_1' \cup A$, with $U_1 \setminus \{0\}$ monochromatic in color 1.

□

The proof of Theorem 6 is essentially the same, except that we use Theorem 11 in place of Theorem 10.

We now reformulate the off-diagonal Ramsey problem as an affine extremal problem. For a subset $A \subseteq \mathbb{F}_2^n$, let $\omega(A)$ be the maximum t such that $A \cup \{0\}$ contains a linear t-space. Define the *sumset* of A to be the set $A + A := \{x + y : x, y \in A\}$, and let $\mathcal{B}_t = \{B \subseteq \mathbb{F}_2^m : m \geq 1, \omega(B + B) \geq t\}$ for $t \geq 1$. Define $m(t)$ to be the minimum n such that $\mathrm{ex}_{\mathrm{aff}}(\mathcal{B}_t) < 2^{n-t+1}$. The following observation is implicit in the work of Nelson and Nomoto [14] on the structural characterization of claw-free binary matroids.

Lemma 12 (Nelson, Nomoto)**.** *For any $t \geq 2$, $R_2(2, t) \leq m(t)$.*

Nelson and Nomoto used the following result of Sanders [17] to prove Theorem 4.

Theorem 13 (Sanders)**.** *Let A be a subset of $\mathbb{F}_2^n$ of density $\alpha < 1/2$. Then*

$$\omega(A + A) \geq n - \left\lceil n / \log_2 \frac{2 - 2\alpha}{1 - 2\alpha} \right\rceil .$$

The proof of Theorem 4 from [14] is simply an application of Theorem 13 with $\alpha = 2^{1-t}$ and $n = (t+1)2^t$, noting that $n - \lceil n / \log_2 \frac{2-2\alpha}{1-2\alpha} \rceil \geq \alpha n/2 - 1 = t$ for this choice of parameters, so $m(t) \leq n$. By Lemma 12, $R_2(2, t) \leq n$ as well.

We observe that the same bound can be obtained by simply applying Theorem 10 instead of Sanders' result, noting that any set A which properly contains an affine $(t-1)$-space has $\omega(A + A) \geq t$, and hence

$$\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_t) \leq \mathrm{ex}_{\mathrm{aff}}(n, \mathbb{F}_2^{t-1}) < 2^{(1-2^{2-t})n+1}.$$

This implies by Lemma 12 that $R_2(2, t) \leq m(t) \leq t2^{t-2}$.

The same argument, together with Theorem 11, gives Theorem 7 for $\mathbb{R}_3(2, t)$. In place of the sumset $A + A$, we consider a set of the form

$$A_{\rightarrow} := \{d \in \mathbb{F}_3^n : \text{there exists } x \text{ such that } x + \lambda d \in A \text{ for all } \lambda \in \mathbb{F}_3\}.$$

To improve on this initial bound for $m(t)$, we consider additional affine structures beyond $\mathbb{F}_2^{t-1}$ that belong to the family $\mathcal{B}_t$. By taking products of smaller structures which have a certain supersaturation property, we construct a sequence $(B_t)_{t \geq 4}$ with $B_t \in \mathcal{B}_t$ and $\mathrm{ex}_{\mathrm{aff}}(n, B_t) < (2^{1-c6^{-t/4}})^n$ for some absolute constant c. This implies Theorem 5, as we have

$$R_2(2, t) \leq m(t) \leq \frac{1}{c}(t-1)6^{t/4}.$$

We leave out most of the details of our argument for the sake of brevity, but we outline our methods. We construct $B_t \in \mathcal{B}_t$ as follows. For $k \geq 2$, define $C_{2k} = \{e_1, \ldots, e_{2k-1}, \sum_{i=1}^{2k-1} e_i\} \subseteq \mathbb{F}_2^{2k-1}$, where e_i is the ith standard basis vector. We observe that $\omega(C_6 + C_6) = 4$. We further observe that for any $A \subseteq \mathbb{F}_2^n$ and $B \subseteq \mathbb{F}_2^m$, the Cartesian product $\{(x, y) \in \mathbb{F}_2^{n+m} : x \in A, y \in B\}$ satisfies

$$\omega((A \times B) + (A \times B)) = \omega(A + A) + \omega(B + B).$$

Thus taking B_t to be $C_6^{\lceil t/4 \rceil}$ gives $B_t \in \mathcal{B}_t$. We also obtain $\mathrm{ex}_{\mathrm{aff}}(n, B_t) < \left(2^{1-c6^{-t/4}}\right)^n$, as desired, via an iterative process that makes use of supersaturation of C_6, in the spirit of [7].

We believe our bounds on $\mathrm{ex}_{\mathrm{aff}}(n, \mathcal{B}_t)$, and hence on $m(t) \geq R_2(2, t)$, to be far from the truth. It remains an open problem to improve these bounds.

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