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Odd-Sunflowers

(Extended abstract)

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Abstract

Extending the notion of sunflowers, we call a family of at least two sets an *odd-sunflower* if every element of the underlying set is contained in an odd number of sets or in none of them. It follows from the Erdős–Szemerédi conjecture, recently proved by Naslund and Sawin, that there is a constant $\mu < 2$ such that every family of subsets of an n-element set that contains no odd-sunflower consists of at most μ^n sets. We construct such families of size at least 1.5021^n.

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1 Introduction

A family of at least 3 sets is a *sunflower* (or a Δ-*system*) if every element is contained either in all of the sets, or in at most one. If a family of sets contains no sets that form a sunflower, it is called *sunflower-free*. This notion was introduced by Erdős and Rado [10] in 1960, and it has become one of the standard tools in extremal combinatorics [14]. Erdős and Rado conjectured that the maximum size of any sunflower-free family of k-element

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sets is at most c^k, for a suitable constant $c > 0$. This conjecture is still open; for recent progress, see [4].

Erdős and Szemerédi [11] studied the maximum possible size of a sunflower-free family of subsets of $\{1, \ldots, n\}$. Denote this quantity by $f(n)$ and let $\mu = \lim f(n)^{1/n}$. Erdős and Szemerédi conjectured that $\mu < 2$, and this was proved by Naslund and Sawin [18], using the methods of Croot, Lev, P. Pach [6], Ellenberg and Gijswijt [8], and Tao [19]. They showed that $\mu < 1.89$, while the best currently known lower bound, $\mu > 1.551$, follows from a construction of Deuber *et al.* [7].

Several variants of the above notion have also been considered. Erdős, Milner and Rado [9] called a family of at least 3 sets a *weak sunflower* if the intersection of any pair of them has the same size. For a survey, see Kostochka [16]. In the literature, we can also find pseudo-sunflowers [13] and near-sunflowers [3]. By restricting the parities of the sets, other interesting questions can be asked, some of which can be answered by the so-called linear algebra method (even-town, odd-town theorems; see [5]).

We introduce the following new variants of sunflowers.

Definition 1. *A nonempty family of nonempty sets forms an* even-degree sunflower *or, simply, an* even-sunflower*, if every element of the underlying set is contained in an even number of sets (or in none). Analogously, a family of at least two nonempty sets forms an* odd-degree sunflower *or, simply, an* odd-sunflower*, if every element of the underlying set is contained in an odd number of sets, or in none.*

Note that any family of pairwise disjoint sets is an odd-sunflower but not an even-sunflower. A (classical) sunflower is an odd-sunflower if and only if it consists of an odd number of sets. In particular, an odd-sunflower-free family is also sunflower-free, as any sunflower contains a sunflower that consists of 3 sets. on the other hand, there exist many odd-sunflowers that contain no sunflower of size 3. For example, $\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ is a minimal odd-sunflower. This example can be generalized as follows.

Let $\mathcal{C}_n$ denote the $(n-1)$-uniform family consisting of all $(n-1)$-element subsets of $\{1, \ldots, n\}$. Let $\mathcal{C}_n^+$ denote the same family completed with the set $\{1, \ldots, n\}$. Obviously, $\mathcal{C}_n$ is an odd-sunflower if and only if n is even, and it is an even-sunflower if and only if n is odd. The family $\mathcal{C}_n^+$ is an odd-sunflower if and only if n is odd, and it is an even-sunflower if and only if n is even. Notice that in any subfamily of these families the nonzero degrees of the vertices differ by at most one. Therefore, in every subfamily of $\mathcal{C}_n$ and $\mathcal{C}_n^+$ which is odd- or even-suflower, all nonzero degrees need to be the same, showing that $\mathcal{C}_n$ and $\mathcal{C}_n^+$ are minimal odd- or even-sunflowers. There are many other examples; e.g., all graphs in which every degree is odd/even are 2-uniform odd/even-sunflowers. In fact, we can show that it is NP-complete to decide whether an input family is odd-sunflower-free or not, so there is no hope of a characterization. This is in contrast with (classic) sunflowers, where the problem is trivially in P.

The main goal of this paper is to raise the following questions: What is the maximum size of a family $\mathcal{F}$ of subsets of $\{1, \ldots, n\}$ that contains no even-sunflower or no odd-sunflower? We denote these maximums by $f_{even}(n)$ and by $f_{odd}(n)$, respectively. As in the case of the even-town and odd-town theorems, the answers to these questions are quite different.

Theorem 2. *$f_{even}(n) = n$, i.e., for any even-sunflower-free family $\mathcal{F} \subset 2^{\{1,\ldots,n\}}$ we have $|\mathcal{F}| \le n$.*

Theorem 3. *$f_{odd}(n) > 1.502148^n$ if $n > n_0$, i.e., there are odd-sunflower-free families $\mathcal{F} \subset 2^{\{1,\ldots,n\}}$, for any large enough n with $|\mathcal{F}| > 1.502148^n$.*

Let $\mu_{odd} = \lim f_{odd}(n)^{1/n}$. (The existence of the limit easily follows from our Lemma 5 and Fekete's lemma, just like for classical sunflowers; see [1].) Using the fact that any odd-sunflower-free family $\mathcal{F}$ is also sunflower-free, the result of Naslund and Savin [18] mentioned above implies that $f_{odd}(n) \le 1.89^n$. Thus, we have

$$1.502148 < \mu_{odd} \le \mu < 1.89.$$

It would be interesting to decide whether μ_{odd} is strictly smaller than μ, and to find a direct proof for $\mu_{odd} < 2$. Is the new slice rank method required?

Some of the ideas used in the proof of Theorem 3 originate, in a slightly different form, in [2]; see Lemmas 5 and 6, and also the discussion on the MathOverflow website [17]. Here, we use a similar approach to recursively construct large odd-sunflower-free families of subsets of $\{1, \ldots, n\}$.

In Section 3, we will also establish a (negative) structural result: If n is large enough, the largest odd-sunflower-free families on the underlying set $\{1, \ldots, n\}$ cannot be obtained by repeatedly adding a small construction to itself, in a simple way (to be described in Lemma 5). We will refer to this method as the "brick construction."

We end this section with a definition. A family $\mathcal{F}$ is called an *antichain*, or *Sperner*, if it is containment-free, i.e., $F, G \in \mathcal{F}$ and $F \subset G$ imply that $F = G$. Let $f_{oa}(n)$ denote the maximum size of an antichain $\mathcal{F}$ on the underlying set $\{1, \ldots, n\}$ that contains no odd-sunflower. Note that any *slice* of $\mathcal{F}$, i.e., any subfamily of $\mathcal{F}$ whose sets are of the same size, form an antichain. Obviously, we have $f_{odd}(n)/n \le f_{oa}(n) \le f_{odd}(n)$ and, therefore,

$$\lim f_{oa}(n)^{1/n} = \mu_{odd}.$$

2 Proof of Theorem 2

The lower bound $f_{even}(n) \ge n$ follows from taking n singleton sets. For the upper bound $f_{even}(n) \le n$, we sketch the argument in two different forms: using linear algebra (as in the usual proof of the odd-town theorem) and by a parity argument (which does not work there).

First proof. Represent each set by its characteristic vector over $\mathbb{F}_2^n$. If $|\mathcal{F}| > n$, these vectors have a nontrivial linear combination that gives zero. The sets whose coefficients are one in this combination yield an even-sunflower. □

Second proof. There are $2^{|\mathcal{F}|}-1$ nonempty subfamilies of $\mathcal{F}$. If $|\mathcal{F}|>n$, by the pigeonhole principle, there are two different subfamilies that contain precisely the same elements of $\{1,\dots,n\}$ an odd number of times. But then their symmetric difference is an even-sunflower. □

3 Brick Constructions

We start with the following simple construction.

Construction 1: Let $k=\lfloor n/3\rfloor$. Make k disjoint groups of size 3 from $\{1,\dots,n\}$. Define $\mathcal{F}$ as the family of all sets that intersect each group in exactly 2 elements. Then we have $|\mathcal{F}|=3^k$, i.e., $\sqrt[3]{3}^n$, whenever n is divisible by 3. This shows that

$$\mu_{odd} \geq \sqrt[3]{3} > 1.44. \tag{1}$$

To prove that this construction is odd-sunflower-free, we need some simple lemmas.

In a *multifamily* of sets, every set F can occur a positive integer number of times. This number is called the *multiplicity* of F. A multifamily of at least two nonempty sets is an *odd-sunflower* if the degree of every element of the underlying set is odd or zero. Note that, similarly to sunflowers, restricting an odd-sunflower multifamily to a smaller underlying set also gives an odd-sunflower multifamily, unless fewer than two nonempty sets remain.

Lemma 4. *If $\mathcal{F}$ is odd-sunflower-free family, and $\mathcal{H}$ is a multifamily of size at least two, comprised of elements $\mathcal{F}$, then $\mathcal{H}$ is an odd-sunflower multifamily if only if it consists of an odd number of copies of a single member $F\in\mathcal{F}$, and an even number of copies of some subsets of F.*

In particular, if $|\mathcal{H}|$ is even, it cannot be an odd-sunflower.

Remark. If $\mathcal{F}$ is an *antichain*, that is, if no $F\in\mathcal{F}$ has a proper subset that belongs to $\mathcal{F}$, then the multifamily $\mathcal{H}$ is an odd-sunflower if and only if it consists of an odd number of copies of the same set $F\in\mathcal{F}$.

Proof. The "if" part of the statement is obvious.

Assume that $\mathcal{H}$ is an odd-sunflower. Reduce the multifamily $\mathcal{H}$ to a family $\mathcal{H}'$ by deleting all sets of even multiplicity and keeping only one copy of each set whose multiplicity is odd. This does not change the parity of the degree of any vertex.

Suppose that $\mathcal{H}'\subseteq\mathcal{F}$ consists of at least two sets. Since $\mathcal{H}'\subseteq\mathcal{F}$ is odd-sunflower-free, there is an element which is contained in a nonzero even number of sets of $\mathcal{H}'$ and, therefore, in a nonzero even number of sets in the multifamily $\mathcal{H}$. This contradicts our assumption that $\mathcal{H}$ was an odd-sunflower.

If $\mathcal{H}'$ is empty, then any element covered by $\mathcal{H}$ is contained in an even number of sets from $\mathcal{H}'$, thus $\mathcal{H}$ again cannot be an odd-sunflower.

Finally, consider the case when the reduced family $\mathcal{H}'$ consists of a single set $F\in\mathcal{F}$. If all sets in the multifamily $\mathcal{H}$ are copies of F, we are done. Otherwise, there are some other sets $F' \neq F$ participating in $\mathcal{H}$ with even multiplicity. If any such F' has an element that does not belong to F, then this element is covered by a nonzero even number of sets of the multifamily $\mathcal{H}$, contradicting the assumption that $\mathcal{H}$ is an odd-sunflower. Therefore, all such F' are subsets of F, as claimed. $\square$

Lemma 5. *If $\mathcal{F}$ and $\mathcal{G}$ are odd-sunflower-free families, and at least one of them is an antichain, then $\mathcal{F}+\mathcal{G}$ is also odd-sunflower-free. Moreover, if both $\mathcal{F}$ and $\mathcal{G}$ are antichains, then so is $\mathcal{F}+\mathcal{G}$.*

Remark. If none of $\mathcal{F}$ and $\mathcal{G}$ are antichains, then it can happen that $\mathcal{F}+\mathcal{G}$ contains an odd-sunflower. For example, if $\mathcal{F} = \{\{1\},\{1,2\}\}$ and $\mathcal{G} = \{\{3\},\{3,4\}\}$, then $\{\{1,3\},\{1,2,3\}, \{1,3,4\}\}$ is an odd-sunflower.

Proof. The "moreover" part of the statement, according to which $\mathcal{F}+\mathcal{G}$ is an antichain, is trivial.

Suppose for contradiction that $\mathcal{F}+\mathcal{G}$ has a subfamily $\mathcal{H}$ consisting of at least two sets that form an odd-sunflower. Without loss of generality, $\mathcal{G}$ is an antichain.

Assume first that the parts of the sets of $\mathcal{H}$ that come from $\mathcal{G}$ are not all the same. These parts are the restriction of $\mathcal{H}$ to the underlying set of $\mathcal{G}$, so they form a multifamily which is an odd-sunflower. Applying Lemma 4 to this subfamily, it follows that the parts of the sets in $\mathcal{H}$ that come from $\mathcal{F}$ all coincide, contradicting our assumption.

Otherwise, the parts of the sets of $\mathcal{H}$ that come from $\mathcal{G}$ are all the same, in which case the parts that come from $\mathcal{F}$ are all different. But then we can use that $\mathcal{F}$ is sunflower-free. $\square$

Corollary 6. *For any integers $n, m, t > 0$, we have $f_{oa}(n)+f_{oa}(m) \geq f_{oa}(n+m)$, $f_{oa}(tn) \geq t f_{oa}(n)$, $\mu_{odd} \geq f_{oa}(n)^{1/n}$, where f_{oa} is the function introduced at the end of Section 1.*

This follows by repeated application of Lemma 5. We can think of $\mathcal{F}$ as the "building block" of $\mathcal{F}+\mathcal{F}+\cdots+\mathcal{F}$, so such constructions may be referred to as *brick constructions* [7]. When $\mathcal{F} = \mathcal{C}_3$ consists of the two-element subsets of $\{1,2,3\}$, we recover Construction 1. This proves (1).

4 Wreath Product Constructions

In this section, we describe another construction that uses the *wreath product* of two families. This is a common notion in group theory [15], but less common in set theory. It was introduced in the PhD thesis of the first author [12]; see also [17].

Let n, m be positive integers, $\mathcal{F} \subseteq 2^{\{1,\ldots,n\}}$, $\mathcal{G} \subseteq 2^{\{1,\ldots,m\}}$ families of subsets of $N = \{1,\ldots,n\}$ and $M = \{1,\ldots,m\}$, respectively. Take n isomorphic copies $\mathcal{G}_1,\ldots,\mathcal{G}_n$ of $\mathcal{G}$ with pairwise disjoint underlying sets $M_1,\ldots,M_n$. Define the *wreath product* of $\mathcal{F}$ and $\mathcal{G}$, denoted by $\mathcal{F} \wr \mathcal{G}$, on the underlying set $\cup_{i=1}^n M_i$, as follows.

$$\mathcal{F} \wr \mathcal{G} = \{\bigcup_{i\in F} G_i \mid F \in \mathcal{F}, G_i \in \mathcal{G}_i\}.$$

That is, for each $F \in \mathcal{F}$, for every $i \in F$, for every $G_i \in \mathcal{G}_i$, we take the set $\cup_{i\in F} G_i$. We obviously have $|\mathcal{F} \wr \mathcal{G}| = \sum_{F\in\mathcal{F}} |\mathcal{G}|^{|F|}$. Thus, $|\mathcal{F} \wr \mathcal{G}| = |\mathcal{F}||\mathcal{G}|^k$ holds, provided that $\mathcal{F}$ is *k-uniform*, i.e., $|F| = k$ for every $F \in \mathcal{F}$.

Lemma 7. *If $\mathcal{F}$ and $\mathcal{G}$ are odd-sunflower-free families and $\mathcal{G}$ is an antichain, then $\mathcal{F} \wr \mathcal{G}$ is also odd-sunflower-free. Moreover, if $\mathcal{F}$ is also an antichain, then so is $\mathcal{F} \wr \mathcal{G}$.*

Remark. If $\mathcal{G}$ is not an antichain, then it may happen that $\mathcal{F} \wr \mathcal{G}$ contains an odd-sunflower, even if $\mathcal{F}$ was an antichain. For example, if $\mathcal{F} = \{\{1, 2\}\}$ and $\mathcal{G} = \{\{3\}, \{3, 4\}\}$, then the three sets $\{3_1, 3_2\}$,$\{3_1, 3_2, 4_1\}$,$\{3_1, 3_2, 4_2\}$ form an odd-sunflower.

Proof. The "moreover" part of the statement, according to which $\mathcal{F} \wr \mathcal{G}$ is an antichain, is trivial.

We need to show that in any family $\mathcal{H}$ of at least two sets from $\mathcal{F} \wr \mathcal{G}$, there is an element contained in a nonzero even number of sets from $\mathcal{H}$. Consider the multifamily $\mathcal{H}'$ of sets from $\mathcal{F}$, in which the multiplicity of a set F is as large, as many sets of the form $\cup_{i\in F} G_i$ belong to $\mathcal{H}$.

Since $\mathcal{F}$ is sunflower-free, there are two possibilities.

Case A: Some set in the multifamily $\mathcal{H}'$ has multiplicity greater than one.

In this case there exists an element $i \in F$ such that the multifamily of sets from $\mathcal{G}_i$, consisting of the intersections of the sets from $\mathcal{H}$ with M_i, has at least two *distinct* sets. Otherwise, the sets of $\mathcal{H}$ that correspond to the repeated set of $\mathcal{H}'$ would coincide, and $\mathcal{H}$ has no repeated sets. Applying Lemma 4 to the multifamily of sets from $\mathcal{G}_i$ for such an i, we find an element of M_i contained in a nonzero even number of sets from $\mathcal{H}$, as required.

Case B: The multifamily $\mathcal{H}'$ is not an odd-sunflower. That is, there exists an element $i \in \{1, \ldots, n\}$ which is covered by an even number of sets in $\mathcal{H}'$.

This means that $\mathcal{H}$ has a nonzero even number of sets with nonempty intersections with M_i. Thus, applying Lemma 4 to the multifamily of sets from $\mathcal{G}_i$ formed by these nonempty intersections, again we find an element of M_i contained in a nonzero even number of sets from $\mathcal{H}$. This completes the proof. □

Corollary 8. *Let $\mathcal{F}$ is a k-uniform odd-sunflower-free antichain on n elements. Then we have*

$$f_{oa}(nm) \geq |\mathcal{F}|(f_{oa}(m))^k.$$

In particular, $f_{oa}(nm) \geq n(f_{oa}(m))^{n-1}$, for odd n.

The second part of the corollary follows by choosing $\mathcal{F} = \mathcal{C}_n$, the family of all $(n-1)$-element subsets of $\{1, \ldots, n\}$. These families have high uniformity, so they are natural candidates to increase the size of the family fast, because the uniformity k appears in the exponent in Corollary 8. As a simple, concrete application, consider the following.

Construction 2: The family $\mathcal{C}_9 \wr \mathcal{C}_3$ consists of $|\mathcal{C}_9||\mathcal{C}_3|^8 = 9 \cdot 3^8 = 3^{10}$ subsets of a $9 \cdot 3 = 27$-element set. Thus, we have

$$\mu_{odd} \geq |\mathcal{C}_9 \wr \mathcal{C}_3|^{1/27} = 3^{10/27} > 1.502144. \tag{2}$$

Lemma 7 implies that $\mathcal{C}_9 \wr \mathcal{C}_3$ contains no odd-sunflower. Thus, $f_{oa}(27) \geq 3^{10}$, and by Corollary 6, $\mu_{odd} \geq f_{oa}(27)^{1/27}$.

By Corollaries 6 and 8, we get $\mu_{odd} \geq f_{oa}(mn)^{1/mn} \geq (n|\mathcal{G}|^{n-1})^{1/mn}$. Here, to get the best bound, we need to choose n so as to maximize the last expression. Letting $n = x|\mathcal{G}|$, we obtain

$$\mu_{odd} \geq (n|\mathcal{G}|^{n-1})^{1/mn} = (x|\mathcal{G}|^n)^{1/mn} = |\mathcal{G}|^{1/m} x^{1/xm|\mathcal{G}|}.$$

Since $|\mathcal{G}|$ and m are independent of n, this is equivalent to maximizing $x^{1/x}$. A simple derivation shows that the optimal choice is $x = e$, so we need n to be the largest odd integer smaller than $e|\mathcal{G}|$, or the smallest odd integer greater than $e|\mathcal{G}|$. In the case of Construction 2, $3e$ is closest to 9.

The above reasoning also shows that any lower bound $|\mathcal{G}|^{1/m} \leq \mu_{odd}$ that comes from the brick construction using $\mathcal{G}$ as a brick, can be slightly improved by taking $\mathcal{C}_n \wr \mathcal{G}$ for some odd n close to $e|\mathcal{G}|$. For example, if $\mathcal{G} = \mathcal{C}_9 \wr \mathcal{C}_3$ is the 16-uniform family of 3^{10} sets on 27 elements obtained in Construction 2, then we can choose n to be $160511 \approx e3^{10}$.

Construction 3: The family $\mathcal{C}_{160511} \wr (\mathcal{C}_9 \wr \mathcal{C}_3)$ consists of $|\mathcal{C}_{160511}||\mathcal{C}_9 \wr \mathcal{C}_3|^{160510} = 160511 \cdot 3^{1605100}$ subsets of a $160511 \cdot 27 = 4333797$-element set. Thus, we have

$$\mu_{odd} \geq (160511 \cdot 3^{1605100})^{1/4333797} > 1.502148. \tag{3}$$

Of course, the improvement on the lower bound for μ_{odd} is extremely small as the families grow.

Concluding remarks

Here we studied the Erdős–Szemerédi-type sunflower problem for odd-sunflowers. We want to remark that our structural result is also true for (classical) sunflowers, using essentially the same proof. That is, if n is large enough, brick constructions will never be optimal. As far as we know, this result is new. The best currently known examples of Deuber *et al.* [7] use a combination of a brick construction and some other *ad hoc* tricks that do not work for odd-sunflowers.

What about the Erdős–Rado-type sunflower problem, i.e., what is the maximum possible size of an odd-sunflower-free k-uniform set system? We pose the following weakening of Erdős and Rado's conjecture:

Conjecture 9. *The maximum size of any odd-sunflower-free family of k-element sets is at most c^k, for a suitable constant $c > 0$.*

Note that the respective problem does not make sense for even-sunflowers, as any number of disjoint sets is even-sunflower-free.

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