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COP NUMBER OF RANDOM k-UNIFORM HYPERGRAPHS

(EXTENDED ABSTRACT)

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Abstract

The game of *Cops and Robber* is usually played on a graph, in which a group of cops attempt to catch a robber moving along the edges of the graph. The *cop number* of a graph is the minimum number of cops required to win the game. An important conjecture in this area, due to Meyniel, states that the cop number of an n-vertex connected graph is $O(\sqrt{n})$. In 2016, Prałat and Wormald [Meyniel's conjecture holds for random graphs, Random Structures Algorithms. 48 (2016), no. 2, 396–421. MR3449604] showed that this conjecture holds with high probability for random graphs above the connectedness threshold. Moreoever, Łuczak and Prałat [Chasing robbers on random graphs: Zigzag theorem, Random Structures Algorithms. 37 (2010), no. 4, 516–524. MR2760362] showed that on a log-scale the cop number demonstrates a surprising *zigzag* behaviour in dense regimes of the binomial random

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graph $G(n, p)$. In this paper, we consider the game of Cops and Robber on a hypergraph, where the players move along hyperedges instead of edges. We show that with high probability the cop number of the k-uniform binomial random hypergraph $G^k(n, p)$ is $O\left(\sqrt{\frac{n}{k}}\log n\right)$ for a broad range of parameters p and k. As opposed to the case of $G(n, p)$, on a log-scale our upper bound on the cop number arises as the minimum of *two* complementary zigzag curves. Furthermore, we conjecture that the cop number of a connected k-uniform hypergraph on n vertices is $O\left(\sqrt{\frac{n}{k}}\right)$.

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1 Introduction and results

1.1 Motivation

The game of *Cops and Robber* was introduced by Quilliot [18] in 1978 and independently by Nowakowski and Winkler [16] in 1983. It is a two-player game played on a simple connected graph $G = (V, E)$, with one player controlling a set of m cops and the other player controlling a single robber. For convenience, we will sometimes refer to the cops and the robber as *pieces*. At the start of the game, the first player chooses a starting vertex for each of the cops, then the second player chooses a starting vertex for the robber. Subsequently, the players take alternating turns and in each turn a player can move each of their pieces to an adjacent vertex (i.e., the pieces move along the edges of G). Note that more than one cop can simultaneously occupy a single vertex and that not every piece must be moved in every turn. The position of all the pieces is known to both players throughout the game. The cops win if at some point in the game a cop occupies the same vertex as the robber, otherwise the robber wins. As this is a game with full information, for each graph G and each number of initial cops m, one of the two players has a winning strategy. The *cop number* $c(G)$ of a graph G is defined as the minimum number $m \in \mathbb{N}$, such that m cops have a winning strategy on G. The cop number has been extensively studied since the introduction of this game.

Whilst there is a structural characterisation of the graphs with cop number one [16], in general the problem of determining the cop number of a graph is EXPTIME-complete [12], and so research in this area has been focused on bounding the cop number of particular graph classes. For example, a classic result of Aigner and Fromme [2] shows that the cop number of a connected planar graph is at most three. More generally, it is known that the cop number is bounded for any proper minor-closed class of graphs [3], and there has been much research into determining the largest cop number of a graph that can be embedded in a fixed surface [7, 9, 13, 19, 20].

Perhaps the most well-known conjecture in this area is Meyniel's conjecture (communicated by Frankl [10]).

Conjecture 1.1. *Let G be a connected graph on n vertices. Then $c(G) = O(\sqrt{n})$.*

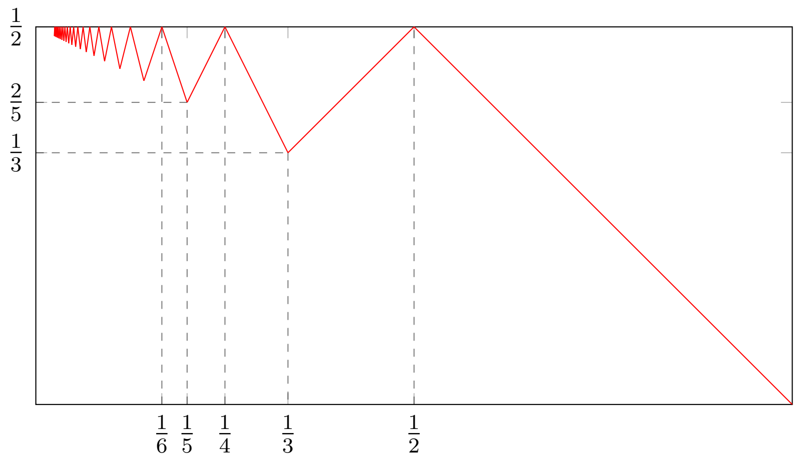

Figure 1: Zigzag shape of the function f

Despite much interest in this conjecture, there has been relatively little improvement to the trivial bound of $O(n)$. Frankl [10] gave the first non-trivial upper bound on the cop number of $O\left(\frac{n \log\log n}{\log n}\right)$, and this bound was improved to $O\left(\frac{n}{\log n}\right)$ by Chiniforooshan [8]. As of today, the best known general upper bound on the cop number is $n2^{-(1+o(1))\sqrt{\log n}}$, given independently by Lu and Peng [14] and by Scott and Sudakov [21]. We note that this bound is still $\Omega\left(n^{1-o(1)}\right)$, and it remains an open question as to whether the cop number can be bounded by $O\left(n^{1-\epsilon}\right)$ for any fixed $\epsilon > 0$ [4].

A natural step towards understanding Conjecture 1.1 is to consider the cop number of the *random* graph $G(n,p)$. For p constant, it was shown by Bonato, Hahn and Wang [6] that with high probability (*whp* for short), meaning with probability tending to one as n tends to infinity, the cop number of $G(n,p)$ is logarithmic in n, and hence Conjecture 1.1 holds for almost all graphs. However, if we let p vary as a function of n, then more interesting behaviour can be seen to develop. Indeed, Łuczak and Prałat [15] showed that whp the function $f\colon (0,1) \to \mathbb{R}$, defined as

$$f(x) = \frac{\log\left(\bar{c}(G\left(n, n^{x-1}\right)\right)}{\log n}, \tag{1.1}$$

where $\bar{c}$ denotes the median of the cop number, has a characteristic zigzag shape (see Figure 1).

Taking the worst case bounds of this function, their result already implies that $c\left(G(n,p)\right) = \tilde{O}\left(\sqrt{n}\right)$ for a broad range of parameters, and that conversely there are choices of p where whp $c(G(n,p)) = \tilde{\Theta}(\sqrt{n})$, where we use $\tilde{\Theta}(\cdot)$ to indicate a bound which holds up to logarithmic factors.

Theorem 1.2 ([15], $\sim$ Theorem 1.1)**.** *Let $\epsilon > 0$, and let $p = \Omega\left(n^{\epsilon-1}\right)$. Then whp*

$$c\left(G(n,p)\right) = \tilde{O}\left(\sqrt{n}\right).$$

In particular, Theorem 1.2 indicates that Conjecture 1.1 holds up to log-factors for this range of p. Bollobás, Kun and Leader [5] gave a similar bound which holds also for sparser regimes of p.

Meyniel's conjecture was finally resolved for all random graphs above the connectedness threshold by Prałat and Wormald [17]. In fact, their result holds for all random graphs with density above $\frac{1}{2}\log n$.

Theorem 1.3 ([17], Theorem 1.2)**.** *Let $\epsilon > 0$, and let $p(n-1) \geq \left(\frac{1}{2}+\epsilon\right)\log n$. Then whp*

$$c\left(G(n,p)\right) = O(\sqrt{n}).$$

In this paper we consider a generalisation of the Cops and Robber game to hypergraphs, and in particular k-uniform hypergraphs, which are called *k-graphs*. The game is defined analogously to the 2-graph case, with the only difference being that the pieces move along hyperedges instead of edges. For the sake of brevity, when it is clear from the context that we are talking about a hypergraph, we will refer to hyperedges as simply edges. Similarly as for 2-graphs, we define for a hypergraph H

$$c\left(H\right) \coloneqq \min\left\{m \in \mathbb{N}\colon m \text{ cops have a winning strategy to catch a robber on } H\right\}.$$

This game was first considered by Gottlob, Leone and Scarcello [11] and by Adler [1]. For more recent results on the hypergraph game we refer the reader to [22], where some classic results on the cop number of 2-graphs are generalised to this setting.

Note that by replacing every edge in the hypergraph by a clique, we arrive at an equivalent 2-graph game. Thus, the game of Cops and Robber on hypergraphs is equivalent to the 2 -graph game played on a restricted class of graphs. On the other hand, we can transform a graph G into a $2k$-uniform hypergraph H with $c\left(G\right) = c\left(H\right)$ via a simple blow-up construction: We replace each vertex v in G by k vertices $\{v_1, v_2, \ldots v_k\}$ and form a hypergraph $H(G)$ on $\{v_i\colon v \in V(G), i \in [k]\}$ by taking an edge of the form $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$ for each edge $e = \{u, v\}$ of G (see Figure 2). It is then easy to check that $c\left(G\right) = c\left(H(G)\right)$, and moreover $|V(H)| = k|V(G)|$.

From these two observations, it is easy to see that the following holds

$$\begin{aligned}\max\left\{c\left(G\right) : G \text{ a graph }, |V(G)| = \frac{2n}{k}\right\} &\leq \max\left\{c\left(H\right) : H \text{ a } k\text{-graph }, |V(H)| = n\right\}\\ &\leq \max\left\{c\left(G\right) : G \text{ a graph }, |V(G)| = n\right\}.\end{aligned}$$

In particular, as there are graphs with $c\left(G\right) = \Omega\left(\sqrt{n}\right)$, there are also k-graphs with $c\left(H\right) = \Omega\left(\sqrt{\frac{n}{k}}\right)$. It would seem surprising that such a simple construction, which is essentially graphical in nature, could capture the worst case behaviour for the cop number in hypergraphs of higher uniformity, but we conjecture that this bound is in fact tight.

Conjecture 1.4. *Let H be a connected k-graph on n vertices. Then $c\left(H\right) = O\left(\sqrt{\frac{n}{k}}\right)$.*

As with Meyniel's Conjecture, a first step towards Conjecture 1.4 would be to consider the behaviour of the cop number of *random* k-graphs.

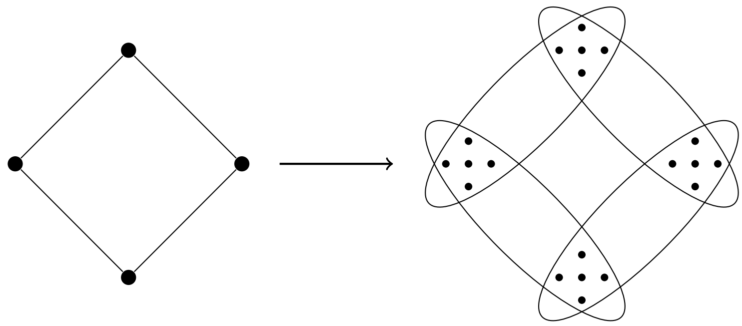

Figure 2: An example of the blow-up construction to generate a $2k$-graph H from a 2-graph that has the same cop number. In this case, $k = 5$, $|V(H)| = 20$ and $c(H) = 2$.

1.2 Main results

The *k-uniform binomial random hypergraph*, which we denote by $G^k(n, p)$, is a random k-graph with vertex set $[n]$ in which each edge, that is, each subset of $[n]$ of size k, appears independently with probability p. Although the main focus of this paper is $G^k(n, p)$, the strategies we develop for the cops will in fact work in a more general class of k-graphs, those satisfying certain expansion properties.

Very roughly, if we denote by $N_V^r(v)$ the vertices that are at most at a fixed distance r from v, then in $G^k(n, p)$ we expect this set to be growing exponentially quickly in r, with its size tightly concentrated around its expectation. Furthermore, for different vertices v and w we do not expect the neighbourhoods $N_V^r(v)$ and $N_V^r(w)$ to have a large intersection, and so, for small subsets $A \subseteq [n]$ we expect the number of vertices at most at a fixed distance r from A to be around $|A|$ times the size of $N_V^r(v)$. Similarly, we expect the set of edges $N_E^r(v)$ at most at a fixed distance $r - 1$ from v to be growing at some uniform exponential rate, and for ranges of p where the random hypergraph is sparse enough, and so few pairs of edges have a large intersection, this rate of growth should be roughly $\frac{1}{k}$ times that of the vertex-neighbourhoods.

Informally, given $\xi > 0$ we will say that a graph is ξ-expanding if the sizes of its vertex and edge-neighbourhoods have this uniform exponential growth, up to some multiplicative error in terms of ξ.

Our first result supports Conjecture 1.4 up to a log-factor for k-graphs that are ξ-expanding for a fixed expansion constant ξ.

Theorem 1.5. *Let $k \geq 2$, let $\xi > 0$ and let G be a ξ-expanding k-graph on n vertices. Then*

$$c(G) \leq 20\xi^{-2}\sqrt{\frac{n}{k}}\log n.$$

Next, we show that whp $G^k(n, p)$ satisfies the desired expansion properties for a broad range of parameters.

Theorem 1.6. *There exists a universal constant ξ such that if $k(n), p(n) > 0$ are such that $k = \omega(\log n)$ and $\frac{n}{k} \geq p\binom{n-1}{k-1} = \omega(\log^3 n)$, then whp $G^k(n,p)$ is ξ-expanding .*

More specifically, after taking a sensible parameterisation, our upper bound on $c(G^k(n,p))$ shows somewhat interesting behaviour, similar to Figure 1. Let us define $\hat{d} = pk\binom{n-1}{k-1}$, which can be thought of as the expected size of the neighbourhood of a vertex in $G^k(n,p)$ and let $\hat{d} = n^\alpha$ and $k = n^\beta$ for some $0 < \beta \leq \alpha \leq 1$. Let us consider the function $f_\beta \colon (\beta, 1) \to \mathbb{R}$ defined as

$$f_\beta(\alpha) = \frac{\log\left(\bar{c}\left(G^k(n,p)\right)\right)}{\log n},$$

with $\bar{c}$ being the upper bound of the cop number as obtained by our strategies (see Section 1.3). Then, f_β again has a characteristic zigzag shape, see Figure 3. In contrast to the case of $G(n,p)$ [15] (see Figure 1), the zigzag shape in the hypergraph case arises as the intersection of two complementary zigzags, coming from two different strategies, and so has twice as many peaks and troughs. We note that under the reasonable assumption that $G^k(n,p)$ is connected, it follows that $d \geq k$, which is the reason as to why the graph in Figure 3 is cut off at $x = \beta$.

We note that, perhaps surprisingly, if we fix k and n and vary d, in certain regimes increasing the average degree, and hence the number of edges, can help the cops, and in other regimes increasing the number of edges can hinder the cops.

Moreover, note that Figure 3 is ignoring log-factors, and in particular, that the worst case bounds, attained when both zigzag lines meet, are of order $O\left(\sqrt{\frac{n}{k}}\log n\right)$. It would be interesting to see, if the log-factor could be removed using similar strategies, and thus show that whp Conjecture 1.4 holds for $G^k(n,p)$.

1.3 Techniques

To give a lower bound for the cop number we need to exhibit a strategy for the cops. As in the work of Łuczak and Prałat [15] we show the existence of a strategy for the cops to *surround* the robber using a probabilistic argument. Whilst in [15] the strategies focused solely on surrounding a small *vertex*-neighbourhood of the robber, we also consider a second type of strategy which aims to surround a small *edge*-neighbourhood, and utilise both these strategies in our result.

Assuming the robber starts on a vertex v, after his first r moves the robber has to be in the r-th vertex-neighbourhood $N_V^r(v)$, and specifically in some edge of the r-th edge-neighbourhood $N_E^r(v)$. The cops aim to occupy each edge in $N_E^r(v)$ before the robber has had time to leave this set. Since the cops move first and a cop can catch the robber in a single move once they occupy the same edge, the cops need to occupy each edge in $N_E^r(v)$ within their first r moves. The strategy of surrounding via vertices, which was used in the 2-graph case by Łuczak and Prałat [15], works quite similarly with the only difference being that the cops surround the r-th vertex-neighbourhood and have $r+1$ moves before the robber can escape. The pay-off in choosing to surround via vertices or edges can be seen as

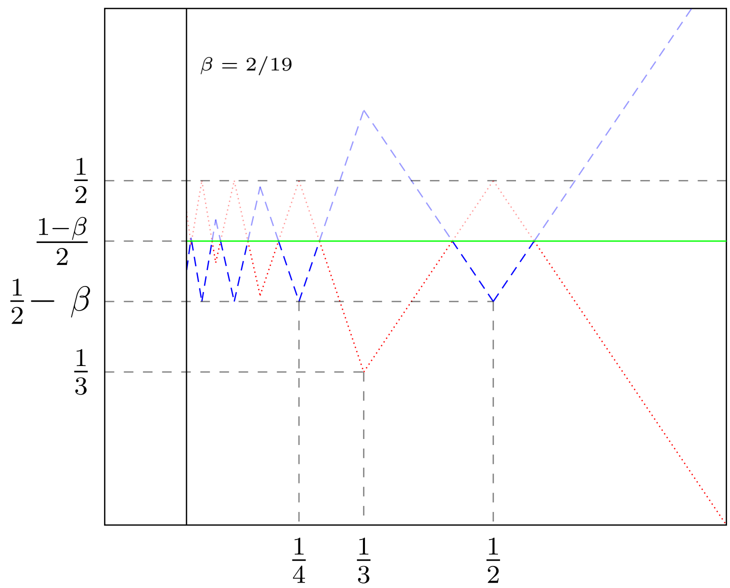

Figure 3: The blue (dashed) line is the upper bound coming from the edge strategy, the red (dotted) line is the upper bound coming from the vertex strategy. As can be seen, the two strategies give rise to two alternating zigzag shapes, that together make up the single zigzag with increased frequency. We note the worst bounds occur at the intersection points of the two lines, which all lie on the green (solid) line at $\frac{1-\beta}{2}$, where β here is equal to $\frac{2}{19}$.

follows – in the former we can use cops at a larger distance, and so in general we will have more cops to work with, whereas in the latter, since each edge contains many vertices, we will not have to occupy as many edges as we would have vertices, and so perhaps we can make do with fewer cops.

For a fixed vertex v and a fixed distance r, the existence of such a strategy can then be reduced to a *matching problem* – for instance in the case of the edge strategy, for each edge e at distance at most r from v we need to assign a unique cop at distance at most r from e, whose strategy will be to occupy e within the first r turns of the game. We aim to show that such an assignment of cops can be found with *positive* probability if we choose a *random* set of cops, assigning a cop to each vertex in the graph independently with some probability $q = q(r)$.

Assuming that our k-graph G is ξ-expanding for some constant $\xi > 0$, we have quite good control over the sizes of $N_V^r(v)$ and $N_E^r(v)$, and also over the number of vertices at a fixed distance from each vertex and edge contained in these sets. Using some standard probabilistic and combinatorial tools, we can show that for an appropriate choice of $q(r)$ with positive probability we can find an appropriate assignment of cops for *each* possible starting vertex v, and bound the number of cops $m(r)$ we use in such a strategy,which in general will depend not only on r, but also on the uniformity k and average degree d of G.

This leads to a family of bounds on the cop number, one for each $r \in \mathbb{N}$, for both the vertex and edge surrounding strategy. For a fixed choice of parameters k and d, we then have to solve an integer optimisation problem to find which choice of r (and of a vertex or edge surrounding strategy) leads to the best bound on the cop number, from which we can derive the bounds leading to Figure 3.

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