

MAXIMUM GENUS ORIENTABLE EMBEDDINGS FROM CIRCUIT DECOMPOSITIONS OF DENSE EULERIAN GRAPHS AND DIGRAPHS

(EXTENDED ABSTRACT)

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Abstract

Suppose we have an eulerian (di)graph with a (directed) circuit decomposition. We show that if the (di)graph is sufficiently dense, then it has an orientable embedding in which the given circuits are facial walks and there are exactly one or two other faces. This embedding has maximum genus subject to the given circuits being facial walks. When there is only one other face, it is necessarily bounded by an euler circuit. Thus, if the numbers of vertices and edges have the same parity, a sufficiently dense (di)graph D with a given (directed) euler circuit C has an orientable embedding with exactly two faces, each bounded by an euler circuit, one of which is C . The main theorem encompasses several special cases in the literature, for example, when the digraph is a tournament.

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1 Introduction and main results

When can a graph or digraph be cellularly embedded in an orientable surface so that it has exactly two faces, each bounded by an euler circuit, such as shown in Figure 1? Is it possible to specify one of the euler circuits in advance? When is it possible to specify

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an arbitrary circuit decomposition of the edges and complete it to an embedding with just one more face, noting that the face is then necessarily bounded by an euler circuit? Finding such a face achieves a maximum genus embedding having the circuits in a given decomposition as facial walks.

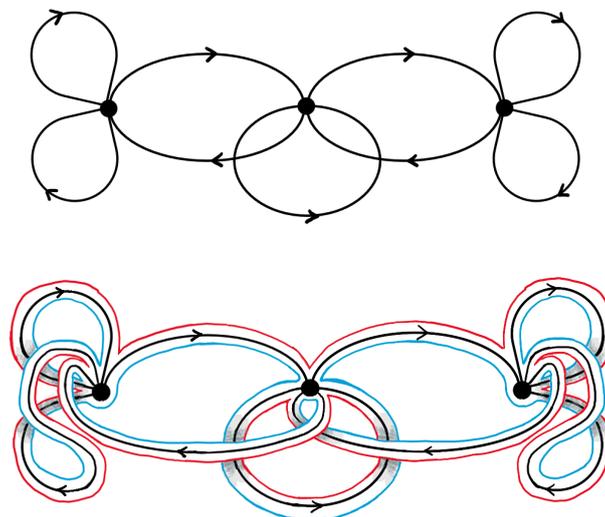


Figure 1: An eulerian digraph (top), and a bi-eulerian orientable embedding of it, showing the two faces bounded by euler circuits, one red and one blue (bottom). The diagram depicts the embedding as a ribbon graph, while sewing a disc into each facial walk gives the surface.

This leads more generally to the question of determining the maximum genus of an embedding relative to a circuit decomposition. Beyond topological graph theory, these questions arise in surprisingly diverse settings, including DNA self-assembly, Steiner triple systems, and latin squares.

Our main result, given in Theorem 1.1, is that if an eulerian (di)graph is sufficiently dense, then it is indeed always possible to achieve these special embeddings of maximum orientable genus.

We allow graphs and digraphs to have loops and multiple edges. A *circuit* in a graph is a closed trail. A graph is *eulerian* if it has a circuit (an *euler circuit*) that uses every edge and every vertex (so the graph is necessarily connected). An embedding of a digraph D is *directed* if every face is bounded by a closed directed walk of D . A *direction-indifferent* embedding of a digraph need not have consistently directed facial walks. In a directed embedding of a digraph, *profaces* are faces whose directed facial walks agree with the clockwise orientation of the surface, while the directed facial walks of the *antifaces* oppose it. We say that an embedding of a (di)graph G is *bi-eulerian* if it has exactly two faces, each bounded by an euler circuit.

Theorem 1.1. *Let D be an n -vertex eulerian digraph where the minimum degree of the underlying simple undirected graph of D is at least $(4n + 2)/5$. Let \mathcal{C} be a directed circuit decomposition of D . Then there is an orientable directed embedding of D with the elements of \mathcal{C} as the profaces and with exactly one or two antifaces, depending on whether $|V(D)| + |A(D)| + |\mathcal{C}|$ is odd or even, respectively.*

This embedding has maximum genus among all orientable directed embeddings of D in which all elements of \mathcal{C} are profaces. This embedding also has maximum genus among all orientable direction-indifferent embeddings of D in which all elements of \mathcal{C} are faces.

A number of related results for both graphs and digraphs follow immediately from this theorem, including Corollary 1.2.

Corollary 1.2. *Let G be an n -vertex eulerian graph where the minimum degree of the underlying simple graph of G is greater than or equal to $(4n + 2)/5$. Let T be any euler circuit in G . Then there is a 2-face-colorable orientable embedding of G with T as the unique face of one color and with exactly one or two faces of the other color, depending on whether $|V(G)| + |E(G)|$ is even or odd, respectively.*

This embedding has maximum genus among all 2-face-colorable orientable embeddings of G . When $|V(G)| + |E(G)|$ is even this is an orientable bi-eulerian embedding of G with T as a specified face, and the embedding has maximum genus among all orientable embeddings of G .

When G additionally has $|V(G)| + |E(G)|$ even, Corollary 1.2 says there is a maximum orientable genus embedding of G that is bi-eulerian. However, in general not every maximum genus embedding of such a G is bi-eulerian.

We do not know if the bound of $(4n + 2)/5$ in Theorem 1.1 is tight. The original motivation for this problem came from an applied problem in DNA self-assembly posed by Jonoska, Seeman and Wu [8], which required a special closed walk in a graph for a DNA strand to follow. A formalization of the walk requirements in [3] led to *edge-outer embeddings* of a graph, that is, orientable embeddings of a graph in which there is a special face whose boundary uses every edge at least once. While [8] proves the existence of such circuits, and [3] gives short algorithmic proof of existence, and proves the hardness of finding an optimal (shortest outer facial walk) solution, there is no control over the number or sizes of the remaining faces in the embedding.

The startlingly simple (to state!) and intriguing questions in the first paragraph emerged from this application. Although determining the size of optimal edge-outer faces is hard in general, for eulerian graphs any optimal edge-outer face is necessarily bounded by an euler circuit. Thus, we seek to control the remaining faces in an optimal edge-outer embedding of an eulerian graph by specifying them in advance with a circuit decomposition. Of particular interest are bi-eulerian orientable embeddings, particularly when one of the circuits is specified in advance.

While our original motivation was DNA self-assembly, these and some closely related questions have also received considerable attention in various other special settings.

In [2], Edmonds proved that every eulerian graph has an bi-eulerian embedding in some surface, but noted that his main theorem was not sufficient to determine the orientability of the embedding. Restricting to the orientable case makes the existence problem quite challenging. From a different perspective, Andersen, Bouchet, and Jackson in [1] focus specifically on compatible euler circuits (A-trails) in 4-regular graphs and digraphs in low genus surfaces.

Furthermore, a series of papers, [4, 5, 6, 7] authored by Griggs and Širáň and sometimes also Erskine, Grannell, McCourt, or Psomas, discusses upper embeddings relative to a triangular decomposition of a graph or digraph, and more specifically completing such a decomposition to an embedding by adding an euler circuit. They are interested in triangular decompositions which arise from structures in design theory such as Steiner triple systems, configurations, and latin squares. One representative theorem can be stated as follows:

Theorem 1.3 (Griggs, McCourt, and Širáň [6, Theorem 1.1]). *Let \mathcal{C} be an oriented Steiner triple system, i.e., a decomposition of a regular tournament D into directed triangles. Then there is an orientable directed embedding of D with the elements of \mathcal{C} as the profaces and with exactly one directed euler circuit antiface.*

The results in these papers are generally specific to triangular decompositions of very special classes of (di)graphs, such as complete or complete tripartite graphs. Our results for arbitrary dense graphs encompass their results that involve complete graphs or tournaments, such as Theorem 1.3.

2 Manipulating facial circuits and proof outline

We first note that we can always form *some* orientable embedding of a digraph with a given circuit decomposition as the profaces. The new faces are necessarily antifaces. We then develop a number of structural results for manipulating the facial circuits in embedded graphs. The following lemmas for example allow us to combine antifaces without altering the given profaces.

A central tool is the following version for directed embeddings of a well-known operation on embeddings of undirected graphs. It allows us to combine three antifaces incident with the same vertex into a single face.

Lemma 2.1. *Let Φ be an orientable directed embedding of an eulerian digraph D , and $v \in V(D)$. If A , B , and C are distinct antifaces that each contain v , then there is an orientable directed embedding Φ' of D that has the same profaces and antifaces as Φ except that A , B and C are merged into a single antiface.*

Interlaced vertices on a circuit, that is, a pair of vertices u and v that appear as $\dots u \dots v \dots u \dots v$ in the circuit, play a pivotal role here. When we have appropriately interlaced vertices, we can merge faces, as in Lemma 2.2.

Lemma 2.2. *Let Φ be an orientable directed embedding of an eulerian digraph D , and $x, y \in V(D)$. Suppose that there are three distinct antifaces A, B, C where x and y are interlaced on A , x occurs on B , and y occurs on C . Then there is an orientable directed embedding Φ' of D that has the same profaces and antifaces as Φ except that A, B and C are replaced by a single antiface.*

The density of the given (di)graph plays an important role in assuring the necessary interlacements. Given a digraph D , let k be such that for each $v \in V(D)$ there are at most k vertices different from v that are not adjacent to v in D . A *locally irreducible embedding* has at most two antifaces meeting at any vertex (which can be guaranteed by repeated use of Lemma 2.1), and a vertex has type AB if it lies on both of the faces A and B . We write $|AB|$ for the number of vertices of type AB . The following is a representative lemma assuring interlacement.

Lemma 2.3. *Suppose that we have a locally irreducible embedding with distinct antifaces A and B . Suppose that $|AB| \geq \min(3k + 4, 4k + 3)$ and there exist vertices of type AP and vertices of type BQ with $P, Q \notin \{A, B\}$ (possibly $P = Q$). Then there is a vertex of type AB that is either interlaced on A with a vertex of type AP or interlaced on B with a vertex of type BQ .*

To prove Theorem 1.1, we use results such as those given above to successively reduce the number of antifaces in the orientable directed embedding of the digraph without changing the given profaces.

The proof begins by applying results such as Lemma 2.2 so that no more than two antifaces are incident with any one vertex. We can then create a *touch graph*, K . The vertices of K are the antifaces. Each vertex v of D gives an edge of K : if v is incident with only one antiface the edge is a loop at that vertex of K , and if v is incident with exactly two antifaces the edge joins the corresponding vertices of K .

We use the structure of the touch graph K , in particular the existence and location of its loops, to organize the necessary case work for the proof of Theorem 1.1. In each situation density information allows us to apply results such as Lemma 2.3 to reduce the number of antifaces. By repeated reductions we obtain an orientable embedding with one or two antifaces and the prescribed set of profaces. If there is only one antiface it must be bounded by a directed euler circuit.

Although our focus here has been on dense graphs, there are certainly also sparse graphs that have bi-eulerian orientable embeddings, as for example in Figure 1. Theorem 2.4 below has a constructive proof with an algorithm that, given an eulerian digraph with a directed euler circuit, produces a second eulerian circuit for the desired bi-eulerian embedding. On the other hand, we have examples of 4-edge-connected, 4-regular graphs with no bi-eulerian embeddings.

Theorem 2.4. *Let D be an eulerian digraph with all vertices of degree congruent to 2 mod 4, and let T be any directed euler circuit of D . Then D has a bi-eulerian orientable embedding with one of the faces bounded by T .*

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