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Precoloring extension in planar near-Eulerian-triangulations*

(Extended abstract)

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Abstract

We consider the 4-precoloring extension problem in *planar near-Eulerian- triangulations*, i.e., plane graphs where all faces except possibly for the outer one have length three, all vertices not incident with the outer face have even degree, and exactly the vertices incident with the outer face are precolored. We give a necessary topological condition for the precoloring to extend, and give a complete characterization when the outer face has length at most five and when all vertices of the outer face have odd degree and are colored using only three colors.

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1 Introduction

Recall that a k*-coloring* of a graph G is a mapping using k colors such that adjacent vertices receive different colors and that a graph is *Eulerian* if all of its vertices have even degree. We study the precoloring extension problem for planar (near) Eulerian triangulations, in particular from an algorithmic perspective.

Famously, the Four Color Theorem states that all planar graphs are 4-colorable [1] and thus from an algorithmic point of view, the problem of determining if a planar graph is 4-colorable is trivial. In contrast, deciding if a planar graph is 3-colorable is a well known

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NP-complete problem [20]. If we move to graphs on surfaces, the situation becomes less clear. Recall that a graph G is $(k+1)$-*critical* if all proper subgraphs of G are k-colorable, but G itself is not. Thus, $(k+1)$-critical graphs are exactly the minimal forbidden subgraphs for k-colorability. A deep result of Thomassen [22] says that for any fixed surface Σ, there are only finitely many $(k+1)$-critical graphs for $k \geq 5$, and combined with a result of Eppstein [7], this implies that there is a polynomial time algorithm for k-coloring graphs on any fixed surface for any $k \geq 5$. Unfortunately, we cannot extend this to 4-colorablity—it is known that there are infinitely many 5-critical graphs on any surface other than the plane [22]. This is a consequence of the folowing result of Fisk [9]: If G is a triangulation of an orientable surface and G has exactly two vertices u and v of odd degree, then u and v must have the same color in any 4-coloring of G, and thus the graph $G + uv$ is not 4-colorable. Even though there are infinitely many 5-critical graphs, it is an important open question if for any fixed surface Σ, there is a polynomial time algorithm to decide if a graph drawn on Σ is 4-colorable.

Let us remark that we have a positive answer to a similar question in the case of 3-coloring triangle-free graphs on surfaces. It is known that there are infinitely many 4-critical triangle-free graphs on all surfaces other than the plane, yet there is a linear time algorithm to decide if a triangle-free graph on any fixed surface is 3-colorable [6]. The algorithm consists of two parts: In the first part, the problem is reduced to (near) quadrangulations [5], and the second part gives a topological criterion for 3-colorability of near quadrangulations [2].

Our hope is that (near) Eulerian triangulations could play the same intermediate role in the case of 4-colorability of graphs on surfaces. Indeed, there is a number of arguments and analogies supporting this idea:

(A) The only constructions of "generic" (e.g., avoiding non-trivial small separations) non-4-colorable graphs drawn on a fixed surface that we are aware of are based on near Eulerian triangulations, such as Fisk's construction [9] or adding vertices to faces of non-3-colorable quadrangulations of the projective plane [13].

(B) As noted above, quadrangulations play key role in the problem of 3-colorability of triangle-free graphs on surfaces, which is a bit surprising at a glance since planar quadrangulations are actually 2-colorable. Analogously, planar Eulerian triangulations are 3-colorable (and in fact, a planar graph is 3-colorable if and only if its a subgraph of a planar Eulerian triangulation).

(C) Many results for quadrangulations of surfaces correspond to results for Eulerian triangulations. As an example, Youngs [25] famously proved that a graph drawn in the projective plane so that all faces have even length is 3-colorable if and only if it does not contain a non-bipartite quadrangulation as a subgraph. For an Eulerian triangulation T of the projective plane, Fisk [8] showed that T has an independent set U such that all faces of $T - U$ have even length, and Mohar [13] proved that T is 4-colorable if and only if $T - U$ is 3-colorable.

As another example, Hutchinson [11] proved that every graph drawn on a fixed orientable surface with only even-length faces and with sufficiently large *edgewidth* (the length of the shortest non-contractible cycle) is 3-colorable, and Nakamoto et al. [16] and Mohar and Seymour [14] have shown that such graphs on non-orientable surfaces are 3-colorable unless they contain a quadrangulation with an odd-length orienting cycle. Analogously, for any orientable surface, any Eulerian triangulation with sufficiently large edgewidth is 4-colorable [12], and for non-orientable surfaces, the only non-4-colorable Eulerian triangulations of large edgewidth are those that have an independent set whose removal results in an even-faced non-3-colorable graph [15].

In this paper, we make the first step towards towards the design of a polynomial-time algorithm to decide whether an Eulerian triangulation of a fixed surface is 4-colorable. In particular, we give the following algorithm. A *planar near-Eulerian-triangulation* is a plane graph where all the faces except possibly for the outer one have length three and all the vertices not incident with the outer face have even degree.

Theorem 1. *There is a linear-time algorithm that given*

- *a planar near-Eulerian-triangulation G with the outer face bounded by a cycle C such that all vertices of C have odd degree in G, and*
- *a precoloring φ of the vertices incident with the outer face of G using only three colors,*

decides whether φ extends to a 4-coloring of G.

The motivation for considering the special case of planar near-Eulerian-triangulations with precolored outer face comes from the general approach towards solving problems for graphs on surfaces, which can be seen e.g. in [2, 3, 4, 18, 19, 21, 23], as well as many other works and is explored systematically in the hyperbolic theory of Postle and Thomas [17]. The general outline of this approach is as follows:

- Generalize the problem to surfaces with boundary, with the boundary vertices precolored (or otherwise constrained).
- Use this generalization to reduce the problem to "generic" instances (e.g., those without short non-contractible cycles, since if an instance contains a short non-contractible cycle, we can cut the surface along the cycle and try to extend all the possible precolorings of the cycle in the resulting graph drawn in a simpler surface).
- The problem is solved in the basic case of graphs drawn in a disk and on a cylinder (plane graphs with one or two precolored faces).
- Finally, the general case is solved with the help of the two basic cases (reducing it to the basic cases by further cutting the surface and carefully selecting the constraints on the boundary vertices [2, 19], using quantitative bounds from the basic cases to show that truly generic cases do not actually arise [17]).

Thus, Theorem 1 is a step towards solving the basic case of graphs drawn in a disk. It unfortunately does not solve this case fully because of the extra assumption that φ only uses three colors (and the assumption that vertices of C have odd degree). Without this assumption, we were able to solve the problem when the precolored outer face has length at most five.

Theorem 2. *There is a linear-time algorithm that given*

- *a planar near-Eulerian-triangulation G with the outer face bounded by a cycle C of length at most five and*
- *a precoloring φ of C*

decides whether φ extends to a 4-coloring of G.

In general, we were only able to find a necessary topological condition for such an extension to exist (which we do not discuss in this extended abstract). We conjecture that using this topological condition in combination with further ideas, it will be possible to resolve the disk case in full.

Conjecture 3. *For every positive integer ℓ, there is a polynomial-time algorithm that given*

- *a planar near-Eulerian-triangulation G with the outer face of length at most ℓ and*
- *a precoloring φ of the vertices incident with the outer face*

decides whether φ extends to a 4-coloring of G.

For the remaining part of the paper, we sketch some of the ideas needed to prove Theorem 1 and Theorem 2.

2 Proving Theorems 1 and 2

Our goal in this section is to show the precoloring extension problem is equivalent to special homomorphisms to the infinite triangular grid equipped with colorings, and then show that in the special cases of Theorems 1 and 2 we can decide if such a homomorphism exists. We need some definitions.

A *hued graph* is a graph G together with a proper 3-coloring $\psi_G : V(G) \to \mathbb{Z}_3$. We will also need to fix a 4-coloring of G, and for notational convenience we use elements of $\mathbb{Z}_2^2$ as colors. With this, a *dappled graph* is a hued graph G together with a proper 4-coloring $\varphi_G : V(G) \to \mathbb{Z}_2^2$. For a vertex $v \in V(G)$, we say that $\psi_G(v)$ is the *hue* and $\varphi_G(v)$ is the *color* of v. *Homomorphisms* of dappled graphs are required to preserve edges and both hue and color, i.e., $f : V(G) \to V(H)$ is a homomorphism if $f(u)f(v) \in E(H)$ for every $uv \in E(G)$ and $\psi_H(f(v)) = \psi_G(v)$ and $\varphi_H(f(v)) = \varphi_G(v)$ for every $v \in V(G)$.

The *dappled triangular grid* is the infinite dappled graph $\mathbf{T}$ with vertex set $\{(i, j) : i, j \in \mathbb{Z}\}$, where vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $(i_2 - i_1, j_2 - j_1) \in$

$\{\pm(1,0), \pm(0,1), \pm(1,1)\}$, with vertex hue $\psi_{\mathbf{T}}(i,j) = (i+j) \bmod 3$ for each vertex (i,j), and with vertex color $\varphi_{\mathbf{T}}(i,j) = (i \bmod 2, j \bmod 2)$ for each vertex (i,j).

The following claim follows by an inspection of the definition.

Observation 4. *Let* $\mathbf{T}$ *be the dappled triangular grid. For any vertex* $v \in V(\mathbf{T})$*, if* u_1 *and* u_2 *are distinct neighbors of* v*, then* $(\psi_{\mathbf{T}}(u_1), \varphi_{\mathbf{T}}(u_1)) \neq (\psi_{\mathbf{T}}(u_2), \varphi_{\mathbf{T}}(u_2))$*. Consequently, for any connected dappled graph* G *and any vertex* $x \in V(G)$*, if* $f_1, f_2 : V(G) \to V(\mathbf{T})$ *are homomorphisms and* $f_1(x) = f_2(x)$*, then* $f_1 = f_2$*.*

Let us now give the key property of dappled patches; the proof easily follows from the coloring-flow duality of Tutte [24].

Theorem 5. *Every dappled patch* G *has a homomorphism to the dappled triangular grid* $\mathbf{T}$*.*

A 4-coloring φ of a connected hued graph H is *viable* if and only if the dappled graph H^{φ} where φ is the associated 4-coloring has a homomorphism to the dappled triangular grid. Let us note that by Observation 4, this condition is easy to verify, as such a homomorphism is unique up to the arbitrary choice of the image of a single vertex of C.

Corollary 6. *Let* G *be a hued patch and* φ *a* 4*-coloring of the boundary of the outer face of* G*. If* φ *extends to a* 4*-coloring of* G*, then* φ *is viable.*

With this, we now sketch how to prove Theorem 1 and 2. The key observation is that in both cases, the homomorphism to the dappled triangular grid $\mathbf{T}$ associated with the precoloring φ maps C to the closed neighborhood of a single vertex of $\mathbf{T}$.

A *hexagon* is a dappled subgraph of $\mathbf{T}$ induced by a vertex and its neighbors. A 4-coloring φ of a connected hued graph C is a *single-hexagon coloring* if it is viable and the corresponding homomorphism f maps C^{φ} to a subgraph of a hexagon H of $\mathbf{T}$. The *central hue* c and the *central color* k of a single-hexagon coloring is the hue and the color of the central vertex of H. There are two important examples of single-hexagon colorings, corresponding to the assumptions of Theorems 1 and 2, respectively.

- Let us call a patch *odd* if its outer face is bounded by a cycle C and all vertices incident with the outer face have odd degree. Every coloring of C that uses at most three colors is single-hexagon, with the central hue 2 and central color not appearing on C.
- Every viable 4-coloring of a hued (≤ 5)-cycle is single-hexagon.

A *retract* of a hued graph G is an induced subgraph H of G such that there exists a *retraction* f from G to H, i.e., a homomorphism such that $f(v) = v$ for each $v \in V(H)$. The following key observation follows from the fact that each shortest cycle in a bipartite graph is a retract [10]. For a dappled graph H, we let H^{-} refer to the underlying hued graph.

Lemma 7. *For every hexagon H of the triangular grid $\mathbf{T}$, the hued hexagon H^- is a retract of $\mathbf{T}^-$*

This implies that the precoloring extension problem with four colors in hued patches for single-hexagon precolorings can be reduced to the problem of 3-precoloring extension in bipartite graphs.

Corollary 8. *Let G be a hued patch and let $\varphi : V(C) \to \mathbb{Z}_2^2$ be a single-hexagon 4-coloring of the boundary C of the outer face of G, with central hue c and central color k. Let $K = \mathbb{Z}_2^2 \setminus \{k\}$. Let H be the bipartite subgraph of G induced by the vertices of hue different from c, and let $\varphi' : V(H) \to K$ be the restriction of φ to H. Then φ extends to a 4-coloring of G if and only if φ' extends to a 3-coloring of H (using the colors in K).*

Dvořák, Král' and Thomas [6] gave for every b a linear-time algorithm to decide whether a precoloring of at most b vertices of a planar triangle-free graph extends to a 3-coloring. Hence, we have the following consequence, a common strengthening of Theorems 1 and 2.

Corollary 9. *For every ℓ, there exists an algorithm than, given a hued patch G with the outer face of length at most ℓ and a single-hexagon precoloring φ of the boundary of the outer face, decides in linear time whether φ extends to a 4-coloring of G.*

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