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On the minimum number of inversions to make a digraph k-(arc-)strong.

(Extended abstract)

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Abstract

The *inversion* of a set X of vertices in a digraph D consists of reversing the direction of all arcs of $D\langle X\rangle$. We study $\mathrm{sinv}'_k(D)$ (resp. $\mathrm{sinv}_k(D)$) which is the minimum number of inversions needed to transform D into a k-arc-strong (resp. k-strong) digraph and $\mathrm{sinv}'_k(n) = \max\{\mathrm{sinv}'_k(D) \mid D \text{ is a } 2k\text{-edge-connected digraph of order } n\}$. We show : (i) $\frac{1}{2}\log(n-k+1) \leq \mathrm{sinv}'_k(n) \leq \log n + 4k - 3$ for all $n \in \mathbb{Z}_{\geq 0}$; (ii) for any fixed positive integers k and t, deciding whether a given oriented graph $\vec{G}$ satisfies $\mathrm{sinv}'_k(\vec{G}) \leq t$ (resp. $\mathrm{sinv}_k(\vec{G}) \leq t$) is NP-complete ; (iii) if T is a tournament of order at least $2k+1$, then $\mathrm{sinv}'_k(T) \leq \mathrm{sinv}_k(T) \leq 2k$, and $\frac{1}{2}\log(2k+1) \leq \mathrm{sinv}'_k(T) \leq \mathrm{sinv}_k(T)$ for some T; (iv) if T is a tournament of order at least $28k-5$ (resp. $14k-3$), then $\mathrm{sinv}_k(T) \leq 1$ (resp. $\mathrm{sinv}_k(T) \leq 6$); (v) for every $\epsilon > 0$, there exists C such that $\mathrm{sinv}_k(T) \leq C$ for every tournament T on at least $2k+1+\epsilon k$ vertices.

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1 Introduction

Notation not given below is consistent with [7]. In particular, a digraph may contain digons but no loops or parallel arcs and an oriented graph is a digraph without digons. We denote by $[k]$ the set $\{1, 2, \ldots, k\}$.

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A **feedback arc set** in a digraph is a set of arcs whose reversal results in an acyclic digraph. Finding a minimum cardinality feedback arc set is one of the first problems shown to be NP-hard listed by Karp in [18]. Furthermore, it is hard to approximate [17, 12]. For tournaments, the problem remains NP-complete [2, 11], but there is a 3-approximation algorithm [1] and a polynomial-time approximation scheme [19].

To make a digraph D acyclic, one can use a different operation from arc reversal, called inversion. The **inversion** of a set X of vertices consists in reversing the direction of all arcs of $D\langle X\rangle$, the subdigraph induced by X. We say that we **invert** X in D. The resulting digraph is denoted by $\mathrm{Inv}(D; X)$. If $(X_i)_{i\in I}$ is a family of subsets of $V(D)$, then $\mathrm{Inv}(D; (X_i)_{i\in I})$ is the digraph obtained after inverting the X_i one after another. Observe that this is independent of the order in which we invert the X_i : $\mathrm{Inv}(D; (X_i)_{i\in I})$ is obtained from D by reversing the arcs such that an odd number of the X_i contain its two end-vertices. The **inversion number** of an oriented graph D, denoted by $\mathrm{inv}(D)$, is the minimum number of inversions needed to transform D into an acyclic oriented graph. It was first introduced by Belkhechine et al. in [10] and then studied in several papers [6, 23, 4, 3].

The main purpose of this article is to study the possibilities of applying the inversion operation to obtain a different objective than the obtained digraph being acyclic. Instead of making a digraph acyclic, we are interested in making it satisfy a prescribed connectivity property. A digraph D is **strongly connected** or simply **strong** (resp. **k-arc-strong**) for some positive integer k, if for any partition (V_1, V_2) of $V(D)$ with $V_1, V_2 \neq \emptyset$ there is an arc (resp. at least k arcs) with tail in V_1 and head in V_2. For a given digraph D, we denote by $\mathrm{UG}(D)$ the undirected (multi)graph that we obtain by suppressing the orientations of the arcs. A digraph is **k-connected** (resp. **k-edge-connected**) if its underlying (multi)graph is. Clearly, a digraph D can be made k-arc-strong by reversing some arcs if and only if the edges of $\mathrm{UG}(D)$ can be oriented such that the resulting digraph is k-arc-strong. Robbins' Theorem [22] asserts that a graph admits a strong orientation if and only if it is 2-edge-connected, and more generally, Nash–Williams' orientation theorem [21], asserts that a graph admits a k-arc-strong orientation if and only if it is $2k$-edge-connected. It is well known that, by reducing to a minimum-cost submodular flow problem, one can determine, in polynomial time, a minimum set of arcs in D whose reversal gives a k-arc-strong digraph or detect that such a set does not exist, see Section 8.8.4 of [7] for details. The digraphs that contain a linear number of vertices with no outgoing arc show that the number of necessary arc reversals to make a $2k$-edge-connected digraph D k-arc-strong cannot be bounded by a function depending only on k. However this is the case for **tournaments**, which are the orientations of complete graphs: Bang-Jensen and Yeo [5] proved that every tournament on at least $2k + 1$ vertices can be made k-arc-strong by reversing at most $\frac{1}{2}k(k - 1)$ arcs. This result is tight for transitive tournaments.

We are interested in the problem of using inversions to make a digraph k-arc-strong. The **k-arc-strong inversion number** of a digraph D, denoted by $\mathrm{sinv}'_k(D)$, is the minimum number of inversions needed to transform D into a k-arc-strong digraph. We study $\mathrm{sinv}'_k(n) = \max\{\mathrm{sinv}'_k(D) \mid D\ 2k\text{-edge-connected digraph of order } n\}$. For all $n \in \mathbb{Z}_{\geq 0}$, we show

$$\frac{1}{2}\log(n-k+1) \le \mathrm{sinv}'_k(n) \le \log n + 4k - 3.$$

To establish the upper bound, it is enough to consider minimally k-edge-connected digraphs which are k-edge-connected digraphs D such that $D \setminus uv$ is not k-edge connected for any arc uv of D. We show that such a digraph D is d-degenerate for $d = 2k-1$, that is, every subdigraph of D has a vertex of degree at most d. Now a result of a forthcoming paper [16] by a group containing the authors asserts that any orientation $\vec{G}_1$ of an n-vertex d-degenerate graph G can transformed into any other orientation of $\vec{G}_2$ of G by inverting at most $\log n + 2d - 1$ sets. Together with a slight strengthening of Nash-Williams' Theorem, we deduce that $\mathrm{sinv}'_k(D) \le \log n + 2(2k-1) - 1$.

Then, we prove that, for any fixed positive integers k and t, deciding whether a given oriented graph $\vec{G}$ satisfies $\mathrm{sinv}'_k(\vec{G}) \le t$ is NP-complete. The case $t = 1$ is proved using a reduction from MONOTONE EQUITABLE k-SAT. An instance of this problem consists of a set of variables X and a set of clauses $\mathcal{C}$ each of which contains exactly $2k+1$ nonnegated variables and the question is whether there is a truth assignment $\phi : X \to \{\text{true}, \text{false}\}$ such that every clause in $\mathcal{C}$ contains at least k true and k false variables with respect to ϕ. The case $t \ge 2$ is proved using a reduction from k-CUT-COVERING. Given a graph G, this problem consists in deciding whether there is a collection $F_1, \dots, F_t$ of cuts such that $\cup_{i=1}^t F_i = E(G)$. We also show that, unless P=NP, sinv'_k cannot be approximated within a factor better than 2.

One may also want to make a digraph k-strong. A digraph D is **k-strong** if $|V(D)| \ge k+1$ and for any set $S \subseteq V(D)$ with less than k vertices $D - S$ is strong. A digraph which can be made k-strong by reversing arcs is **k-strengthenable**. The 1-strengthenable digraphs are the 2-edge-connected ones, because being 1-strong is equivalent to being strong or 1-arc-strong. Thomassen [24] proved that the 2-strengthenable digraphs are the 4-edge-connected digraphs D such that $D - v$ is 2-edge-connected for every vertex $v \in V(D)$, but it is NP-hard to compute the minimum number of arc reversals needed to make a given digraph 2-strong [8]. Furthermore, in contrast to the analogous problem for k-arc-strengthenable digraphs, for $k \ge 3$, it is NP-complete to decide whether a digraph is k-strengthenable. Indeed, for any $k \ge 3$, it is NP-complete to decide whether an undirected graph has a k-strong orientation [13].

It is also natural to use inversions to make a digraph k-strong. A **k-strengthening family** of a digraph D is a family of subsets $(X_i)_{i\in I}$ of subsets of $V(D)$ such that $\mathrm{Inv}(D; (X_i)_{i\in I})$ is k-strong. The **k-strong inversion number** of a k-strengthenable digraph D, denoted by $\mathrm{sinv}_k(D)$, is the minimum number of inversions needed to transform D into a k-strong digraph. We show that for any positive integers k and t, it is NP-complete to decide whether $\mathrm{sinv}_k(D) \le t$ for a given k-strengthenable oriented graph. We also show that, unless P=NP, sinv_k cannot be approximated within a factor better than 2. The proofs are similar to the ones for sinv'_k.

It is not hard to show that every tournament of order at least $2k+1$ is k-strengthenable and that it can be made k-strong by reversing the orientation of at most $\frac{1}{4}(4k-2)(4k-$

3) arcs, see e.g. [7, p. 379]. In 1994, Bang-Jensen conjectured that every tournament on at least $2k+1$ vertices can be made k-strong by reversing at most $\frac{1}{2}k(k+1)$ arcs. Bang-Jensen, Johansen, and Yeo [9] proved this conjecture for tournaments of order at least $3k-1$. It is then natural to ask whether or not we can make a tournament k-strong or k-arc-strong in a lot less than $\frac{1}{2}k(k+1)$ inversions. This leads to consider $M_k = \max\{\mathrm{sinv}_k(T) \mid T \text{ tournament of order at least } 2k+1\}$ and $M'_k = \max\{\mathrm{sinv}'_k(T) \mid T \text{ tournament of order at least } 2k+1\}$. We show that (for sufficiently large k), we have

$$\frac{1}{2}\log(2k+1) \leq M'_k \leq M_k \leq 2k.$$

The lower bound is obtained for a tournament of order $2k+1$ by using McKay's result [20] on the number of Eulerian tournaments of order $2k+1$, the fact that every k-arc-strong tournament of order $2k+1$ is Eulerian and counting arguments. Let us now prove the upper bound.

Let D be a digraph and u,v two distinct vertices in D. The **strong-connectivity** from u to v in D, denoted by $\kappa_D(u,v)$, is the maximal number α such that $D-X$ contains a (u,v)-path for every $X \subseteq V(D) \setminus \{u,v\}$ with $|X| \leq \alpha - 1$. For some $S \subseteq V(D)$ and positive integer k, we say that S is **k-strong in** D if $\kappa_D(u,v) \geq k$ for all $u,v \in S$. The following statement is well-known.

Lemma 1.1. *Let D be a digraph, S a k-strong set in D and $v \in V(D) \setminus S$. If v has k in-neighbours in S and k out-neighbours in S, then $S \cup \{v\}$ is k-strong in D.*

Theorem 1.2. *$M_k \leq 2k$.*

Proof. Let D be a tournament with $V(D) = \{v_1, \ldots, v_n\}$ with $n \geq 2k+1$. Further, let T be a k-strong tournament on $\{v_1, \ldots, v_{2k+1}\}$. We now define sets $X_1, \ldots, X_{2k}$. Suppose that the sets $X_1, \ldots, X_{i-1}$ have already been created and let D_{i-1} be the graph obtained from D by inverting $X_1, \ldots, X_{i-1}$. Now let $X_i = \{v_i\} \cup A_i \cup B_i$, where A_i is the set of vertices v_j with $j \in \{i+1, \ldots, 2k+1\}$ for which the edge v_iv_j has a different orientation in T and D_{i-1}, and B_i is, when $i \leq k$ (resp. $i \geq k+1$), the set of vertices v_j with $j \geq 2k+2$ for which D_{i-1} contains the arc v_iv_j (resp. v_jv_i).

Observe that $D_{2k}\langle\{v_1, \ldots, v_{2k+1}\}\rangle = T$ which is k-strong by assumption. Moreover, for any $j \geq 2k+2$, D_{2k} contains the arcs v_jv_i for $i \in [k]$ and the arcs v_iv_j for $i = k+1, \ldots, 2k$. Hence, by Lemma 1.1, D_{2k} is k-strong. □

We also prove that $M_1 = M'_1 = 1$ and $M_2 = M'_2 = 2$ showing that the bound $M_k \leq 2k$ is not tight for $k = 1, 2$. We believe that it is also not tight for larger values of k.

It is not too difficult to prove that every sufficiently large tournament can be made k-strong in one inversion. Hence it is natural to investigate the minimum integer $N_k(i)$ (resp. $N'_k(i)$) such that $\mathrm{sinv}_k(T) \leq i$ (resp. $\mathrm{sinv}'_k(T) \leq i$) for every tournament T of order at least $N_k(i)$. We prove

$$5k-2 \leq N_k(1) \leq 28k-5 \text{ and } N_k(6) \leq 14k-3.$$

The lower bound $N_k(1) \geq 5k-2$ is obtained by considering a tournament T of order $5k-3$ whose vertex set has a partition (A, B, C) such that $T\langle A\rangle$ and $T\langle C\rangle$ are $(k-1)$-diregular tournaments of order $2k-1$, and $A \Rightarrow B \cup C$ and $B \Rightarrow C$, and proving $\text{sinv}'_k(T) > 1$.

The upper bounds $N_k(1) \leq 28k-5$ and $N_k(6) \leq 14k-3$ are obtained using median orders. A **median order** of D is an ordering $(v_1, v_2, \dots, v_n)$ of the vertices of D with the maximum number of forward arcs, that arcs v_iv_j with $j > i$. Our proofs use the two well-known properties (M1) and (M2) in the next lamma (the feedback property in [15]), which allow to prove the third one (M3). We denote by $R^+_D(v)$ (resp. $R^-_D(v)$) the set of vertices which are **reachable** from vertex v (resp. from which v can be reached) in digraph D, that are the vertices w such that there is a directed (v, w)-path (resp. (w, v)-path) in D.

Lemma 1.3. *Let T be a tournament and $(v_1, v_2, \dots, v_n)$ a median order of T. Then, for any two indices i, j with $1 \leq i < j \leq n$:*

(M1) *$(v_i, v_{i+1}, \dots, v_j)$ is a median order of the induced subtournament $T\langle\{v_i, v_{i+1}, \dots, v_j\}\rangle$.*

(M2) *v_i dominates at least half of the vertices $v_{i+1}, v_{i+2}, \dots, v_j$, and v_j is dominated by at least half of the vertices $v_i, v_{i+1}, \dots, v_{j-1}$. In particular, each vertex v_i, $1 \leq i < n$, dominates its successor v_{i+1}.*

(M3) *For any $X \subseteq V(T) \setminus \{v_i\}$, $|R^+_{T-F}(v_i)| \geq n+1-i-2|F|$, and $R^-_{T-F}(v_i) \geq i-2|F|$.*

Let T be a tournament of order $n \geq 28k-5$ and let $(v_1, \dots, v_n)$ be a median order of $V(T)$. Let $A = \{v_{n-6k+1}, \dots, v_n\}$ and $B = \{v_1, \dots, v_{6k}\}$. Using Lemma 1.3, we show that there is a set $X \subseteq A \cup B$ such that in the tournament $T_0 = \text{Inv}(T\langle A \cup B\rangle, X)$, for any $Y \subseteq V(T_0)$ with $|Y| \leq k-1$, there is a directed path from a to $B \setminus Y$ in $T_0 - Y$ for every $a \in A \setminus Y$, and there is a directed path from $A \setminus Y$ to b in $T_0 - Y$ for every $b \in B \setminus Y$. We then show that $\text{Inv}(T, X)$ is k-strong. This proves $N_k(1) \leq 28k-5$.

The fact that there exists a constant $\alpha > 0$ such that every tournament on at least αk vertices can be made k-strong by a single inversion raises the following question: for which $\alpha > 2$, every tournament on at least αk vertices can be made k-strong by a constant number of inversion? We show that every $\alpha > 2$ will do : there is a function f such that for every $\epsilon > 0$ and $k \in \mathbb{N}$, $\text{sinv}_k(T) \leq f(\epsilon)$ for every tournament T on at least $2k+1+\epsilon k$ vertices.

The proof is based on a probabilistic argument: we show that $f(\epsilon)$ inversions drawn uniformly at random, under the constraint that they cover all the vertices, make such a tournament k-strong with high probability.

Finally, the fact that $m_k(n) = 1$ for n sufficiently large (in comparison to k) implies that the set $\mathcal{F}_k$ of tournaments T such that $\text{sinv}_k(T) > 1$ is finite. This implies that for fixed k computing sinv_k and sinv'_k can be done in polynomial time for tournaments.

The proofs of the results announced in this extended abstract can be found in the full version of the paper [14].

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