On the minimum number of inversions to make a digraph $k$-(arc-)strong.

(Extended abstract)

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Abstract

The inversion of a set $X$ of vertices in a digraph $D$ consists of reversing the direction of all arcs of $D(X)$. We study $\text{sinv}'_k(D)$ (resp. $\text{sinv}_k(D)$) which is the minimum number of inversions needed to transform $D$ into a $k$-arc-strong (resp. $k$-strong) digraph and $\text{sinv}'_k(n) = \max\{\text{sinv}'_k(D) \mid D$ is a $2k$-edge-connected digraph of order $n\}$. We show: (i) $\frac{1}{2}\log(n-k+1) \leq \text{sinv}'_k(n) \leq \log n + 4k - 3$ for all $n \in \mathbb{Z}_{\geq 0}$; (ii) for any fixed positive integers $k$ and $t$, deciding whether a given oriented graph $\vec{G}$ satisfies $\text{sinv}'_k(\vec{G}) \leq t$ (resp. $\text{sinv}_k(\vec{G}) \leq t$) is NP-complete; (iii) if $T$ is a tournament of order at least $2k+1$, then $\text{sinv}'_k(T) \leq \text{sinv}_k(T) \leq 2k$, and $\frac{1}{2}\log(2k+1) \leq \text{sinv}'_k(T) \leq \text{sinv}_k(T)$ for some $T$; (iv) if $T$ is a tournament of order at least $28k - 5$ (resp. $14k - 3$), then $\text{sinv}_k(T) \leq 1$ (resp. $\text{sinv}_k(T) \leq 6$); (v) for every $\epsilon > 0$, there exists $C$ such that $\text{sinv}_k(T) \leq C$ for every tournament $T$ on at least $2k+1+\epsilon k$ vertices.

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1 Introduction

Notation not given below is consistent with [7]. In particular, a digraph may contain digons but no loops or parallel arcs and an oriented graph is a digraph without digons. We denote by $[k]$ the set $\{1, 2, \ldots, k\}$.  

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A feedback arc set in a digraph is a set of arcs whose reversal results in an acyclic digraph. Finding a minimum cardinality feedback arc set is one of the first problems shown to be NP-hard by Karp in [13]. Furthermore, it is hard to approximate [17] [12]. For tournaments, the problem remains NP-complete [2] [11], but there is a 3-approximation algorithm [11] and a polynomial-time approximation scheme [19].

To make a digraph $D$ acyclic, one can use a different operation from arc reversal, called inversion. The inversion of a set $X$ of vertices consists in reversing the direction of all arcs of $D(X)$, the subdigraph induced by $X$. We say that we invert $X$ in $D$. The resulting digraph is denoted by $\text{Inv}(D; X)$. If $(X_i)_{i \in I}$ is a family of subsets of $V(D)$, then $\text{Inv}(D; (X_i)_{i \in I})$ is the digraph obtained after inverting the $X_i$ one after another. Observe that this is independent of the order in which we invert the $X_i$ : $\text{Inv}(D; (X_i)_{i \in I})$ is obtained from $D$ by reversing the arcs such that an odd number of the $X_i$ contain its two end-vertices.

The inversion number of an oriented graph $D$, denoted by $\text{inv}(D)$, is the minimum number of inversions needed to transform $D$ into an acyclic oriented graph. It was first introduced by Belkhechine et al. in [10] and then studied in several papers [6] [23] [4] [3].

The main purpose of this article is to study the possibilities of applying the inversion operation to obtain a different objective than the obtained digraph being acyclic. Instead of making a digraph acyclic, we are interested in making it satisfy a prescribed connectivity property. A digraph $D$ is strongly connected or simply strong (resp. $k$-arc-strong) for some positive integer $k$, if for any partition $(V_1, V_2)$ of $V(D)$ with $V_1, V_2 \neq \emptyset$ there is an arc (resp. at least $k$ arcs) with tail in $V_1$ and head in $V_2$. For a given digraph $D$, we denote by $\text{UG}(D)$ the undirected (multi)graph that we obtain by suppressing the orientations of the arcs. A digraph is $k$-connected (resp. $k$-edge-connected) if its underlying (multi)graph is. Clearly, a digraph $D$ can be made $k$-arc-strong by reversing some arcs if and only if the edges of $\text{UG}(D)$ can be oriented such that the resulting digraph is $k$-arc-strong. Robbins’ Theorem [22] asserts that a graph admits a strong orientation if and only if it is 2-edge-connected, and more generally, Nash–Williams’ orientation theorem [21], asserts that a graph admits a $k$-arc-strong orientation if and only if it is $2k$-edge-connected. It is well known that, by reducing to a minimum-cost submodular flow problem, one can determine, in polynomial time, a minimum set of arcs in $D$ whose reversal gives a $k$-arc-strong digraph or detect that such a set does not exist, see Section 8.8.4 of [7] for details. The digraphs that contain a linear number of vertices with no outgoing arc show that the number of necessary arc reversals to make a $2k$-edge-connected digraph $D$ $k$-arc-strong cannot be bounded by a function depending only on $k$. However this is the case for tournaments, which are the orientations of complete graphs: Bang-Jensen and Yeo [5] proved that every tournament on at least $2k + 1$ vertices can be made $k$-arc-strong by reversing at most $\frac{1}{2} k(k - 1)$ arcs. This result is tight for transitive tournaments.

We are interested in the problem of using inversions to make a digraph $k$-arc-strong. The $k$-arc-strong inversion number of a digraph $D$, denoted by $\text{inv}'_k(D)$, is the minimum number of inversions needed to transform $D$ into a $k$-arc-strong digraph. We study $\text{inv}'_k(n) = \max \{ \text{inv}'_k(D) \mid D 2k$-edge-connected digraph of order $n \}$. For all $n \in \mathbb{Z}_{\geq 0}$, we
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show
\[ \frac{1}{2} \log(n - k + 1) \leq \sinv_k(n) \leq \log n + 4k - 3. \]

To establish the upper bound, it is enough to consider minimally $k$-edge-connected digraphs which are $k$-edge-connected digraphs $D$ such that $D \setminus uv$ is not $k$-edge connected for any arc $uv$ of $D$. We show that such a digraph $D$ is $d$-degenerate for $d = 2k - 1$, that is, every subdigraph of $D$ has a vertex of degree at most $d$. Now a result of a forthcoming paper by a group containing the authors asserts that any orientation $\vec{G}_1$ of an $n$-vertex $d$-degenerate graph $G$ can transformed into any other orientation of $\vec{G}_2$ of $G$ by inverting at most $\log n + 2d - 1$ sets. Together with a slight strengthening of Nash-Williams’ Theorem, we deduce that $\sinv_k(D) \leq \log n + 2(2k - 1) - 1$.

Then, we prove that, for any fixed positive integers $k$ and $t$, deciding whether a given oriented graph $\vec{G}$ satisfies $\sinv_k(\vec{G}) \leq t$ is NP-complete. The case $t = 1$ is proved using a reduction from MONOTONE EQUITABLE $k$-SAT. An instance of this problem consists of a set of variables $X$ and a set of clauses $C$ each of which contains exactly $2k + 1$ nonnegated variables and the question is whether there is a truth assignment $\phi : X \to \{true, false\}$ such that every clause in $C$ contains at least $k$ true and $k$ false variables with respect to $\phi$. The case $t \geq 2$ is proved using a reduction from $k$-Cut-Covering. Given a graph $G$, this problem consists in deciding whether there is a collection $F_1, \ldots, F_t$ of cuts such that $\cup_{i=1}^t F_i = E(G)$. We also show that, unless $P=NP$, $\sinv'_k$ cannot be approximated within a factor better than 2.

One may also want to make a digraph $k$-strong. A digraph $D$ is $k$-strong if $|V(D)| \geq k + 1$ and for any set $S \subseteq V(D)$ with less than $k$ vertices $D - S$ is strong. A digraph which can be made $k$-strong by reversing arcs is $k$-strengthenable. The 1-strengthenable digraphs are the 2-edge-connected ones, because being 1-strong is equivalent to being strong or 1-arc-strong. Thomassen [24] proved that the 2-strengthenable digraphs are the 4-edge-connected digraphs $D$ such that $D - v$ is 2-edge-connected for every vertex $v \in V(D)$, but it is NP-hard to compute the minimum number of arc reversals needed to make a given digraph 2-strong [8]. Furthermore, in contrast to the analogous problem for $k$-arc-strengthenable digraphs, for $k \geq 3$, it is NP-complete to decide whether a digraph is $k$-strengthenable. Indeed, for any $k \geq 3$, it is NP-complete to decide whether an undirected graph has a $k$-strong orientation [13].

It is also natural to use inversions to make a digraph $k$-strong. A $k$-strengthening family of a digraph $D$ is a family of subsets $(X_i)_{i \in I}$ of subsets of $V(D)$ such that $\text{Inv}(D; (X_i)_{i \in I})$ is $k$-strong. The $k$-strong inversion number of a $k$-strengthenable digraph $D$, denoted by $\sinv_k(D)$, is the minimum number of inversions needed to transform $D$ into a $k$-strong digraph. We show that for any positive integers $k$ and $t$, it is NP-complete to decide whether $\sinv_k(D) \leq t$ for a given $k$-strengthenable oriented graph. We also show that, unless $P=NP$, $\sinv'_k$ cannot be approximated within a factor better than 2. The proofs are similar to the ones for $\sinv_k$.

It is not hard to show that every tournament of order at least $2k + 1$ is $k$-strengthenable and that it can be made $k$-strong by reversing the orientation of at most $\frac{1}{4}(4k - 2)(4k -
3) arcs, see e.g. [1] p. 379]. In 1994, Bang-Jensen conjectured that every tournament on at least \(2k+1\) vertices can be made \(k\)-strong by reversing at most \(\frac{1}{2}k(k+1)\) arcs. Bang-Jensen, Johansen, and Yeo [9] proved this conjecture for tournaments of order at least \(3k-1\). It is then natural to ask whether or not we can make a tournament \(k\)-strong or \(k\)-arc-strong in a lot less than \(\frac{1}{2}k(k+1)\) inversions. This leads to consider \(M_k = \max\{\sinv_k(T) \mid T \text{ tournament of order at least } 2k+1\}\) and \(M'_k = \max\{\sinv'_k(T) \mid T \text{ tournament of order at least } 2k+1\}\). We show that (for sufficiently large \(k\)), we have

\[
\frac{1}{2} \log(2k+1) \leq M'_k \leq M_k \leq 2k.
\]

The lower bound is obtained for a tournament of order \(2k+1\) by using McKay’s result [20] on the number of Eulerian tournaments of order \(2k+1\), the fact that every \(k\)-arc-strong tournament of order \(2k+1\) is Eulerian and counting arguments. Let us now prove the upper bound.

Let \(D\) be a digraph and \(u, v\) two distinct vertices in \(D\). The strong-connectivity from \(u\) to \(v\) in \(D\), denoted by \(\kappa_D(u, v)\), is the maximal number \(\alpha\) such that \(D - X\) contains a \((u, v)\)-path for every \(X \subseteq V(D) \setminus \{u, v\}\) with \(|X| \leq \alpha - 1\). For some \(S \subseteq V(D)\) and positive integer \(k\), we say that \(S\) is \(k\)-strong in \(D\) if \(\kappa_D(u, v) \geq k\) for all \(u, v \in S\). The following statement is well-known.

**Lemma 1.1.** Let \(D\) be a digraph, \(S\) a \(k\)-strong set in \(D\) and \(v \in V(D) \setminus S\). If \(v\) has \(k\) in-neighbours in \(S\) and \(k\) out-neighbours in \(S\), then \(S \cup \{v\}\) is \(k\)-strong in \(D\).

**Theorem 1.2.** \(M_k \leq 2k\).

**Proof.** Let \(D\) be a tournament with \(V(D) = \{v_1, \ldots, v_n\}\) with \(n \geq 2k+1\). Further, let \(T\) be a \(k\)-strong tournament on \(\{v_1, \ldots, v_{2k+1}\}\). We now define sets \(X_1, \ldots, X_{2k}\). Suppose that the sets \(X_1, \ldots, X_{i-1}\) have already been created and let \(D_{i-1}\) be the graph obtained from \(D\) by inverting \(X_1, \ldots, X_{i-1}\). Now let \(X_i = \{v_i\} \cup A_i \cup B_i\), where \(A_i\) is the set of vertices \(v_j\) with \(j \in \{i+1, \ldots, 2k+1\}\) for which the edge \(v_iv_j\) has a different orientation in \(T\) and \(D_{i-1}\), and \(B_i\) is, when \(i \leq k\) (resp. \(i \geq k+1\)), the set of vertices \(v_j\) with \(j \geq 2k+2\) for which \(D_{i-1}\) contains the arc \(v_iv_j\) (resp. \(v_jv_i\)).

Observe that \(D_{2k}(\{v_1, \ldots, v_{2k+1}\}) = T\) which is \(k\)-strong by assumption. Moreover, for any \(j \geq 2k+2\), \(D_{2k}\) contains the arcs \(v_iv_i\) for \(i \in [k]\) and the arcs \(v_iv_j\) for \(i = k+1, \ldots, 2k\). Hence, by Lemma 1.1 \(D_{2k}\) is \(k\)-strong.

We also prove that \(M_1 = M'_1 = 1\) and \(M_2 = M'_2 = 2\) showing that the bound \(M_k \leq 2k\) is not tight for \(k = 1, 2\). We believe that it is also not tight for larger values of \(k\).

It is not too difficult to prove that every sufficiently large tournament can be made \(k\)-strong in one inversion. Hence it is natural to investigate the minimum integer \(N_k(i)\) (resp. \(N'_k(i)\)) such that \(\sinv_k(T) \leq i\) (resp. \(\sinv'_k(T) \leq i\)) for every tournament \(T\) of order at least \(N_k(i)\). We prove

\[5k - 2 \leq N_k(1) \leq 28k - 5\] and \(N_k(6) \leq 14k - 3\).
The lower bound $N_k(1) \geq 5k - 2$ is obtained by considering a tournament $T$ of order $5k - 3$ whose vertex set has a partition $(A, B, C)$ such that $T(A)$ and $T(C)$ are $(k - 1)$-irregular tournaments of order $2k - 1$, and $A \Rightarrow B \cup C$ and $B \Rightarrow C$, and proving $\sinv_k(T) > 1$.

The upper bounds $N_k(1) \leq 28k - 5$ and $N_k(6) \leq 14k - 3$ are obtained using median orders. A median order of $D$ is an ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of $D$ with the maximum number of forward arcs, that arcs $v_iv_j$ with $j > i$. Our proofs use the two well-known properties (M1) and (M2) in the next lemma (the feedback property in [15]), which allow to prove the third one (M3). We denote by $R_D^+(v)$ (resp. $R_D^-(v)$) the set of vertices which are reachable from vertex $v$ (resp. from which $v$ can be reached) in digraph $D$, that are the vertices $w$ such that there is a directed $(v, w)$-path (resp. $(w, v)$-path) in $D$.

**Lemma 1.3.** Let $T$ be a tournament and $(v_1, v_2, \ldots, v_n)$ a median order of $T$. Then, for any two indices $i, j$ with $1 \leq i < j \leq n$:

(M1) $(v_i, v_{i+1}, \ldots, v_j)$ is a median order of the induced subtournament $T\{v_i, v_{i+1}, \ldots, v_j\}$.

(M2) $v_i$ dominates at least half of the vertices $v_{i+1}, v_{i+2}, \ldots, v_j$, and $v_j$ is dominated by at least half of the vertices $v_i, v_{i+1}, \ldots, v_{j-1}$. In particular, each vertex $v_i, 1 \leq i < n$, dominates its successor $v_{i+1}$.

(M3) For any $X \subseteq V(T) \setminus \{v_1\}$, $|R_{T-}^+(v_i)| \geq n + 1 - i - 2|F|$, and $R_{T-}^-(v_i) \geq i - 2|F|$.

Let $T$ be a tournament of order $n \geq 28k - 5$ and let $(v_1, \ldots, v_n)$ be a median order of $V(T)$. Let $A = \{v_{n-6k+1}, \ldots, v_n\}$ and $B = \{v_1, \ldots, v_{6k}\}$. Using Lemma 1.3, we show that there is a set $X \subseteq A \cup B$ such that in the tournament $T_0 = \text{Inv}(T(A \cup B), X)$, for any $Y \subseteq V(T_0)$ with $|Y| \leq k - 1$, there is a directed path from $a$ to $b \setminus Y$ in $T_0 - Y$ for every $a \in A \setminus Y$, and there is a directed path from $A \setminus Y$ to $b$ in $T_0 - Y$ for every $b \in B \setminus Y$. We then show that $\text{Inv}(T, X)$ is $k$-strong. This proves $N_k(1) \leq 28k - 5$.

The fact that there exists a constant $\alpha > 0$ such that every tournament on at least $\alpha k$ vertices can be made $k$-strong by a single inversion raises the following question: for which $\alpha > 2$, every tournament on at least $\alpha k$ vertices can be made $k$-strong by a constant number of inversion? We show that every $\alpha > 2$ will do: there is a function $f$ such that for every $\epsilon > 0$ and $k \in \mathbb{N}$, $\sinv_k(T) \leq f(\epsilon)$ for every tournament $T$ on at least $2k + 1 + \epsilon k$ vertices.

The proof is based on a probabilistic argument: we show that $f(\epsilon)$ inversions drawn uniformly at random, under the constraint that they cover all the vertices, make such a tournament $k$-strong with high probability.

Finally, the fact that $m_k(n) = 1$ for $n$ sufficiently large (in comparison to $k$) implies that the set $\mathcal{F}_k$ of tournaments $T$ such that $\sinv_k(T) > 1$ is finite. This implies that for fixed $k$ computing $\sinv_k$ and $\sinv'_k$ can be done in polynomial time for tournaments.

The proofs of the results announced in this extended abstract can be found in the full version of the paper [14].
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References


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