Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

Tower Gaps in Multicolour Ramsey Numbers

(Extended abstract)

Quentin Dubroff * António Girão † Eoin Hurley ‡ Corrine Yap §

Abstract

Resolving a problem of Conlon, Fox, and Rödl, we construct a family of hypergraphs with arbitrarily large tower height separation between their 2-colour and q-colour Ramsey numbers. The main lemma underlying this construction is a new variant of the Erdős–Hajnal stepping-up lemma for a generalized Ramsey number $r_k(t; q, p)$, which we define as the smallest integer n such that every q-colouring of the k-sets on n vertices contains a set of t vertices spanning fewer than p colours. Our results provide the first tower-type lower bounds on these numbers.

DOI: https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-053

1 Introduction

Let $K_n^{(k)}$ denote the complete k-uniform hypergraph on n vertices. We define $r_k(G; q)$ for $k, q \in \mathbb{N}$ as the smallest integer n such that in every q-colouring of $K_n^{(k)}$, there is a monochromatic copy of the hypergraph G. For simplicity when G is $K_t^{(k)}$, we write $r_k(G; q) = r_k(t; q)$. Observe that when $q = 2$, $r_k(G; 2)$ and $r_k(t; 2)$ coincide with the classical Ramsey numbers $r_k(G)$ and $r_k(t)$, and we will denote them as such. One of the

*Department of Mathematics, Rutgers University, Piscataway, NJ, 08854, USA. E-mail: qcd2@math.rutgers.edu
†Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK. E-mail: antonio.girao@maths.ox.ac.uk
‡Unaffiliated. E-mail: eoin.hurley@umail.ucc.ie
§Department of Mathematics, Rutgers University, Piscataway, NJ, 08854, USA. E-mail: corrine.yap@rutgers.edu

most central open problems in Ramsey theory is determining the growth rate of the 3-uniform Ramsey number $r_3(t)$. A famous result of Erdős, Hajnal, and Rado [8] from the 60's shows that there exist constants c and c' such that

$$2^{ct^2} \leq r_3(t) \leq 2^{2^{c't}}.$$

Note that the upper bound is essentially exponential in the lower bound. Despite much attention, this remains the state of the art. Perhaps surprisingly, if we allow four colours instead of two, Erdős and Hajnal (see e.g. [10]) showed that the double-exponential upper bound is essentially correct, i.e. there is a $c > 0$ such that $r_3(t;4) \geq 2^{2^{ct}}$. More recently Conlon, Fox, and Sudakov [4] proved a super-exponential bound with three colours, that is, that there exists $c > 0$ such that $r_3(t;3) \geq 2^{t^{c \log t}}$. Erdős conjectured that the double-exponential bound should hold without using extra colours, offering \$500 dollars for a proof that $r_3(t) \geq 2^{2^{ct}}$ for some constant $c > 0$. Raising the stakes for this conjecture is the ingenious *stepping-up* construction of Erdős and Hajnal (see e.g. [10]), which shows that for all q and $k \geq 3$,

$$r_{k+1}(2t + k - 4; q) > 2^{r_k(t;q)-1}. \tag{1}$$

For the past 60 years, we have used (1) to stack our lower bounds for $r_k(t;q)$ upon that of $r_3(t;q)$, yielding that $r_k(t) \geq T_{k-1}(ct^2)$, where $T_k(x)$, *the tower of height k in x*, is defined by $T_1(x) = x$, $T_{i+1}(x) = 2^{T_i(x)}$. The corresponding upper bounds of $r_k(t) \leq T_k(O(t))$ (see [6, 7, 8]) are once again exponential in the lower bounds, and thus a positive resolution of Erdős's conjecture would be the decisive step in showing that $r_k(t) = T_k(\Theta(t))$ for all $k \geq 3$.

Due to the lack of progress on this central conjecture, it is natural to try to understand just how significant a role the number of colours can play in hypergraph Ramsey numbers and whether or not there could really be such a large difference between $r_3(t)$ and $r_3(t;4)$. One argument in favour of the conjecture is that the reliance on extra colours to prove a double-exponential lower bound may be a technical limitation of the stepping-up construction. This is challenged by a stunning discovery of Conlon, Fox, and Rödl [3] who exhibited an infinite family of 3-uniform hypergraphs called *hedgehogs*, whose Ramsey numbers display strong dependence on the number of colours. Namely, they showed that the 2-colour Ramsey number of hedgehogs is polynomial in their order, while the 4-colour Ramsey number is at least exponential. To understand just how significant a role the number of colours could play they asked the following:

Question 1.1. *For any integer $h \geq 3$, do there exist integers k and q and a family of k-uniform hypergraphs for which the 2-colour Ramsey number grows as a polynomial in the number of vertices, while the q-colour Ramsey number grows as a tower of height h?*

Our main contribution is to answer this in the affirmative. Define the *k-uniform balanced hedgehog* $\hat{H}_t^{(k)}$ with body of order t to be the graph constructed as follows: take a set S of t vertices, called the *body*, and for each subset $X \subset S$ of order $\lceil \frac{k}{2} \rceil$ add a k-edge e with $e \cap S = X$ such that for all $e, f \in E(\hat{H}_t^{(k)})$ we have $e \cap f \subset S$. The hedgehog $H_t^{(k)}$ as defined by Conlon, Fox, and Rödl differs only in that they consider every $X \subset S$ of order $k-1$ rather than $\lceil \frac{k}{2} \rceil$. We observe that for $k = 3$ the two definitions coincide. When the uniformity is clear from the context we shall drop the superscript.

Theorem 1.2. *There exist $c > 0$ and $q : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$ and sufficiently large t, we have*

(a) $r_{2k+1}(\hat{H}_t) \leq t^{k+3}$*, and*

(b) $r_{2k+1}(\hat{H}_t; q(k)) \geq T_{\lfloor c \log_2 \log_2 k \rfloor}(t)$.

To prove this, we provide new stepping-up lemmas for a more general type of hypergraph Ramsey numbers. Let $r_k(G; q, p)$ for $q \geq p$ be the smallest integer n such that in every q-colouring of $K_n^{(k)}$, there is a copy of the hypergraph G whose edges span fewer than p colours. As before, we use $r_k(t; q, p)$ when $G = K_t^{(k)}$ and suppress p when $p = 2$.

A standard application of the first moment method (see e.g. [1]) shows that for any $k, q \in \mathbb{N}$ there exists $c > 0$ such that $r_k(t; q, q) \geq 2^{ct^{k-1}}$ for all $t \in \mathbb{N}$. We note that in the graph case ($k = 2$) the special case of $q = p$ was already investigated by Erdős and Szemerédi [9] in the 70's; in fact, the more general case when $p < q$ is also indirectly discussed. They showed the following rather precise bounds: for all $q \ll t$, $2^{\Omega(t/q)} \leq r_2(t; q, q) \leq q^{O(t/q)}$.

These generalized hypergraph Ramsey numbers were also considered in a special case by Conlon, Fox, and Rödl [3] who asked if there exist an integer q and number $c > 0$ such that $r_3(t; q, 3) \geq 2^{2^{ct}}$. To date, the only nontrivial improvement on the first moment bound has been made by Mubayi and Suk [11] who proved there exists $c > 0$ such that for $q \geq 9$, we have $r_3(t; q, 3) \geq 2^{t^{2+cq}}$ for $t \in \mathbb{N}$ sufficiently large; for all other values of $k, q, p \geq 3$, the random construction is essentially the state of the art. Our knowledge (or lack thereof) is thus summarised by the following bounds for $k, q, p \geq 3$ and sufficiently large $t \in \mathbb{N}$,

$$2^{t^c} \leq r_k(t; q, p) \leq T_k(O(t)),$$

where $c \geq 1$ is allowed to depend on k, q and p. Note that in this case our upper bounds are a staggering tower of height $k - 2$ in the lower bounds.

A related notion called the set-colouring Ramsey number was introduced by Erdős, Hajnal, and Rado in [8] and subsequently studied in [12] and much more recently in [5] and [2]. Borrowing notation from [5], let $R_k(t; q, s)$ denote the minimum number of vertices such that every (q, s)-*set colouring* of $K_n^{(k)}$, that is, a colouring in which each k-set is assigned an element of $\binom{[q]}{s}$, contains a monochromatic $K_t^{(k)}$. Here, monochromatic means the intersection of the colour sets assigned to the edges is nonempty. Observe that certain cases of R_k and r_k coincide. For example, $R_k(t; q, q-1) = r_k(t; q, q)$ and in general, we have the bound

$$r_k(t; q, p) \leq R_k\left(t; \binom{q}{p-1}, \binom{q-1}{p-2}\right).$$

We prove lower bounds on $r_k(t; q, p)$, thus giving lower bounds on certain set-colouring Ramsey numbers. However, we are not able to definitively resolve any questions from [5], due to central gaps in our understanding of hypergraph Ramsey numbers. See Section 2 for more on this.

Our main tool in the proof of Theorem 1.2 is the development of two new stepping-up constructions which yield the first tower-type results of their kind. We show the following three stepping-up statements, listed in order of decreasing strength, with $C_k = \frac{1}{k+1}\binom{2k}{k}$ denoting the k-th Catalan number.

Theorem 1.3. *Let $k, q, p \geq 3$. There exist $c \geq 1$ and t_0 such that for all $t > t_0$,*

(a) if $p \leq C_k - 2$, then $r_{k+1}(t^c; q, p) > 2^{r_k(t;q,p)-1}$,

(b) if $p \leq C_k$, then $r_{k+1}(t^c; 2q + p, p) > 2^{r_k(t;q,p)-1}$, and

(c) if $p \leq k!$, then $r_{2k}(t^c; qp, p) > 2^{r_k(t;q,p)-1}$.

Note that the growth rate in k which is implied by Part (c) (approximately a tower of height $\log_2 k$) of Theorem 1.3 is much smaller than that of Parts (a) and (b) because we can only step up at the cost of doubling the uniformity size. Unfortunately, this does not allow us to answer the question of Conlon, Fox, and Rödl on $r_3(t; q, 3)$, since $C_2 = 2$, but already for $k \geq 4$ we have the following two corollaries:

Corollary 1.4. *For all $k \geq 4$, there is $q \in \mathbb{N}$ and $c > 0$ such that $r_k(t; q, 5) \geq T_{k-1}(t^c)$.*

Corollary 1.5. *For all $k \geq 4$, there is $c > 0$ such that $r_k(t; 3, 3) \geq T_{k-1}(t^c)$.*

Observe that by the second corollary the growth rate of $r_k(t; 3, 3)$ matches the current best lower bounds for $r_k(t)$ up to a polynomial in t. The reason we have an absolute constant c in the exponent is due to the use of an Erdős-Hajnal type result on sequences.

The second main element of our proof connects the problem of avoiding monochromatic balanced hedgehogs to that of avoiding cliques that span few colours. It is a straightforward adaptation of ideas from Conlon, Fox, and Rödl [3].

Lemma 1.6. *Given $k, q, t \in \mathbb{N}$, let $p = \binom{2k+1}{k+1}$ and $q' = \binom{q}{p}$. Then*

$$r_{2k+1}(\hat{H}_t; q', 2) > r_{k+1}(t; q, p + 1) - 1.$$

Using this result along with Part (c) of Theorem 1.3 yields the lower bound in Theorem 1.2(b). It is natural to ask whether one can combine the growth rate in k given by Part (a) of Theorem 1.3 with the ability to impose as many colours as in Part (c). Unfortunately, the condition $p \leq C_k$ prevents us from using Part (a) as the right-hand side because $C_k = \frac{1}{k+1}\binom{2k}{k} < \binom{2k+1}{k+1}$. This is tantalisingly close, if not a little curious, as the dependence on C_k comes from our exact solution to a subsequence avoidance problem. We show that C_k presents a natural barrier in this endeavour. This barrier is made concrete by some new and tight results on the Ramsey theory of sequences, including an Erdős-Hajnal-type result.

2 Moving Forward

Both of our new stepping-up constructions rely on a dichotomy: either we can find many suitable substructures within the δ-sequences (which give rise to many colours) or we must have a long monotonic sequence (which allows us to use induction). Since for every k-edge there are at most $k!$ distinct permutations, our methods fail to give good lower bounds for $r_k(t; q, p)$ whenever $k \ll p$. Even in the simplest case $r_3(t; q, 3)$, we were not able to prove a double exponential lower bound, leaving open the following question of Conlon, Fox, and Rödl on $r_3(t; q, 3)$.

Problem 2.1. *[3, Problem 1] Is there an integer q, a positive constant c, and a q-colouring of the 3-uniform hypergraph on $2^{2^{ct}}$ vertices such that every subset of order t receives at least 3 colours?*

We propose here a much weaker problem than Problem 2.1 which we were not able to resolve. We note that a negative answer would uncover a radical new phenomenon in the Ramsey numbers of hypergraphs.

Problem 2.2. *Does there exist $k \in \mathbb{N}$ such that the following holds? For all $p \in \mathbb{N}$ there exist $q \in \mathbb{N}$ and $c > 0$ such that $r_k(t; q, p) \geq 2^{2^{t^c}}$ for all t sufficiently large.*

A similar but much more ambitious problem was posed in [5].

Problem 2.3. *[5, Problem 6.3] Determine the tower height of $R_k(n; r, r-1) = r_k(n; r, r)$ for all $k \geq 3$ and $r \geq 2$.*

The authors of [5] note the apparent difficulty of Problem 2.3 and ask the following weaker question. Is there a fixed integer c such that $R_k(n; r, r-1) \geq T_{k-c}(n)$ for every $k \geq 3$ and $r \geq 2$? We cannot answer this question, but using Theorem 1.3(a), we can prove $R_k(n; r, r-1)$ is at least a tower of height roughly $k - 0.5 \log_2 r$. Any improvement beyond this bound would likely be very interesting.

We make the following conjecture regarding the Ramsey numbers of k-uniform hedgehogs. This would in particular demonstrate that the 2-colour and q-colour Ramsey numbers of these hedgehogs, unlike those of balanced hedgehogs, do not differ by arbitrarily large tower heights.

Conjecture 2.4. *There is $\ell \in \mathbb{N}$ such that for every positive integer k, for every sufficiently large t,*

$$r_k(H_t^{(k)}) \geq T_{k-\ell}(t).$$

Acknowledgments

We thank Bhargav Narayanan for helpful conversations on this topic and the anonymous referees for their detailed and invaluable feedback, especially regarding the organization of the first two sections. The first author was supported in part by Simons Foundation grant

332622. The first and fourth authors were supported in part by NSF Grant CCF-1814409 and NSF Grant DMS-1800521. The second author was supported by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC-2181/1 - 390900948 (the Heidelberg STRUCTURES Cluster of Excellence) and by EPSRC grant EP/V007327/1. The third author was supported in part by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 428212407.

References

[1] N. Alon and J.H. Spencer. *The Probabilistic Method.* John Wiley & Sons, 2016. 382

[2] Lucas Aragão, Maurício Collares, João Pedro Marciano, Taísa Martins, and Robert Morris. A lower bound for set-colouring Ramsey numbers. *arXiv preprint arXiv:2212.06802*, 2022. 382

[3] D. Conlon, J. Fox, and V. Rödl. Hedgehogs are not color blind. *J. Comb.*, 8(3):475–485, 2017. 381, 382, 383, 384

[4] D. Conlon, J. Fox, and B. Sudakov. Hypergraph Ramsey numbers. *J. Amer. Math. Soc.*, 23(1):247–266, 2010. 381

[5] David Conlon, Jacob Fox, Xiaoyu He, Dhruv Mubayi, Andrew Suk, and Jacques Verstraëte. Set-coloring Ramsey numbers via codes. *arXiv preprint arXiv:2206.11371*, 2022. 382, 384

[6] P. Erdős and A. Hajnal. On Ramsey like theorems, problems and results. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, pages 123–140, 1972. 381

[7] P. Erdős and R. Rado. Combinatorial theorems on classifications of subsets of a given set. *Proc. London Math. Soc.*, 3(1):417–439, 1952. 381

[8] P. Erdős, A. Hajnal, and R. Rado. Partition relations for cardinal numbers. *Acta Math. Acad. Sci. Hung.*, 16:93–196, 1965. 381, 382

[9] P Erdős and A Szemerédi. On a Ramsey type theorem. *Periodica Mathematica Hungarica*, 2(1-4):295–299, 1972. 382

[10] R.L. Graham, B.L. Rothschild, and J.H. Spencer. *Ramsey Theory*, volume 20. John Wiley & Sons, 1990. 381

[11] D. Mubayi and A. Suk. Cliques with many colors in triple systems. *arXiv:2005.03078*, 2020. 382

[12] Xiaodong Xu, Zehui Shao, Wenlong Su, and Zhenchong Li. Set-coloring of edges and multigraph Ramsey numbers. *Graphs and Combinatorics*, 25(6):863–870, 2009. 382