

CHVÁTAL-ERDŐS CONDITION FOR PANCYCLICITY

(EXTENDED ABSTRACT)

Nemanja Draganić* David Munhá Correia* Benny Sudakov*

Abstract

An n -vertex graph is *Hamiltonian* if it contains a cycle that covers all of its vertices and it is *pancyclic* if it contains cycles of all lengths from 3 up to n . A celebrated meta-conjecture of Bondy states that every non-trivial condition implying Hamiltonicity also implies pancyclicity (up to possibly a few exceptional graphs). We show that every graph G with $\kappa(G) > (1+o(1))\alpha(G)$ is pancyclic. This extends the famous Chvátal-Erdős condition for Hamiltonicity and proves asymptotically a 30-year old conjecture of Jackson and Ordaz.

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1 Introduction

The notion of Hamiltonicity is one of most central and extensively studied topics in Combinatorics. Since the problem of determining whether a graph is Hamiltonian is NP-complete, a central theme in Combinatorics is to derive sufficient conditions for this property. A classic example is Dirac's theorem [14] which dates back to 1952 and states that every n -vertex graph with minimum degree at least $n/2$ is Hamiltonian. Since then, a plethora of interesting and important results about various aspects of Hamiltonicity have been obtained, see e.g. [1, 11, 12, 13, 19, 26, 27, 28, 32], and the surveys [21, 30].

Besides finding sufficient conditions for containing a Hamilton cycle, significant attention has been given to conditions which force a graph to have cycles of other lengths. Indeed, *the cycle spectrum of a graph*, which is the set of lengths of cycles contained in

*Department of Mathematics, ETH, Zürich, Switzerland. Research supported in part by SNSF grant 200021_196965.

Emails: {nemanja.draganic,david.munhacanascorreia, benjamin.sudakov}@math.ethz.ch.

that graph, has been the focus of study of numerous papers and in particular gained a lot of attention in recent years [2, 3, 15, 20, 22, 25, 29, 31, 35]. Among other graph parameters, the relation of the cycle spectrum to the minimum degree, number of edges, independence number, chromatic number and expansion of the graph have been studied.

We say that an n -vertex graph is *pancyclic* if the cycle spectrum contains all integers from 3 up to n . In the cycle spectrum of an n -vertex graph, it is usually hardest to guarantee the existence of the longest cycle, i.e. a Hamilton cycle. This intuition was captured in Bondy's famous meta-conjecture [6] from 1973, which asserts that any non-trivial condition which implies Hamiltonicity, also implies pancyclicity (up to a small class of exceptional graphs). As a first example, he proved in [7] an extension of Dirac's theorem, showing that minimum degree at least $n/2$ implies that the graph is either pancyclic or that it is the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$. Further, Bauer and Schmeichel [5], relying on previous results of Schmeichel and Hakimi [34], showed that the sufficient conditions for Hamiltonicity given by Bondy [8], Chvátal [10] and Fan [18] all imply pancyclicity, up to a certain small family of exceptional graphs.

Another classic condition which implies Hamiltonicity is given by the famous theorem of Chvátal and Erdős [11]. It states that if the connectivity of a graph G is at least as large as its independence number, that is, $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian. The pancyclicity counterpart of this result has also been investigated - see, e.g., [4] and the surveys [23, 33]. In fact, in 1990, Jackson and Ordaz [23] conjectured that G must be pancyclic if $\kappa(G) > \alpha(G)$, which if true would confirm Bondy's meta-conjecture for this classical instance. One can use an old result of Erdős [16] to show pancyclicity if $\kappa(G)$ is large enough function of $\alpha(G)$. A first linear bound on $\kappa(G)$ was given only in 2010 by Keevash and Sudakov [25], who showed that $\kappa(G) \geq 600\alpha(G)$ is enough. In this paper, we resolve the conjecture of Jackson and Ordaz asymptotically, by showing that $\kappa(G) > (1 + o(1))\alpha(G)$ is already enough to guarantee pancyclicity.

Theorem 1. *Let $\varepsilon > 0$ and let n be sufficiently large. Then, every n -vertex graph G for which we have $\kappa(G) \geq (1 + \varepsilon)\alpha(G)$ is pancyclic.*

Next we briefly discuss some of the key steps in the proof of this theorem. It will be convenient for us to consider different ranges of cycle lengths whose existence we want to show, and for each range we have a slightly different approach to deal with. But in general, in order to find these different cycle lengths we will combine various tools on shortening/augmenting paths and finding consecutive path lengths between two fixed vertices.

For example, for finding consecutive path lengths we crucially use that since $\kappa(G) > \alpha(G)$, it must be that G contains triangles - moreover, it contains a *path with triangles attached to many of its edges* (see Definition 2), which trivially implies the existence of many consecutive path lengths between the endpoints of such a path. For shortening/augmenting paths, we also introduce new tools. One of them is used to shorten paths using only the minimum degree of the graph (Lemma 6), while another one augments paths using both the independence and connectivity number, and is given in the complete version of the paper. Furthermore, we will also use a novel result proven in [15] using the Gallai-Milgram

theorem, in order to shorten paths using the independence number of the graph. In our paper, we present these tools, together with some other useful results of a similar flavour. The general proof strategy is to find a cycle of appropriate length which consists of two paths, one of which has many edges to which triangles are attached. Then we apply our shortening/augmenting results to the other path. Combining the consecutive path lengths from the first path with the path lengths obtained from the second path we get all possible cycle lengths.

2 Cycles with triangles and path shortening

In this section we will give a taste of the methods we use. We will show two simple results – first we show how to obtain a cycle with many triangles, and second, in Lemma 6 we show how to shorten a path between two vertices only using the minimum degree of the graph. We start with the definition of a cycle with many triangles.

Definition 2. Define the graph C_ℓ^r to be the graph formed by a cycle $v_1v_2 \dots v_lv_1$ of length ℓ with the additional edges $v_1v_3, v_3v_5, \dots, v_{2r-1}v_{2r+1}$ (if $r = 0$, then it is just a cycle of length ℓ). We will refer to this as a cycle of length ℓ with r triangles. Similarly define P_ℓ^r and refer to it as a path of length ℓ with r triangles, where P_0^0 is just a vertex.

The following is an easy starting point for the existence of the graphs C_ℓ^r with appropriate parameters, as subgraphs in graphs G with $\kappa(G) \geq \alpha(G)$.

Lemma 3. Every n -vertex graph G with $\kappa(G) \geq \alpha(G)$ contains a C_l^r for all r such that $0 \leq r \leq \frac{\kappa(G) - \alpha(G)}{2}$ and some l with $l - 2(r + 1) \leq \max\left(\frac{n}{\kappa(G) - 2r + 1}, \frac{n}{\kappa(G) - 1}\right)$. In particular, it contains a P_{2r}^r for all such r .

Proof. We will first show that G must always contain a $P_{2r'}^{r'}$ for $r' := \left\lfloor \frac{\kappa(G) - \alpha(G)}{2} \right\rfloor$ – we construct such a path greedily. Suppose that we have the vertices $v_1v_2v_3 \dots v_{2i+1}$ which form a P_{2i}^{i-1} , so that the edges $v_1v_3, \dots, v_{2i-1}v_{2i+1}$ are also present. Provided that $i < r'$, we can augment this path as follows. Consider the set $S := N(v_{2i+1}) \setminus \{v_1, \dots, v_{2i}\}$ – by assumption, this has size at least $\delta(G) - 2i > \kappa(G) - 2r' \geq \alpha(G)$. Therefore, it must contain an edge $v_{2i+2}v_{2i+3}$. Clearly, $v_{2i+1}v_{2i+2}v_{2i+3}$ forms a triangle and thus, $v_1v_2v_3 \dots v_{2i+1}v_{2i+2}v_{2i+3}$ is a P_{2i+2}^{i+1} . Continuing with this procedure until $i = r'$, gives the desired $P_{2r'}^{r'}$.

Now, fix r with the given condition. If $r = 0$, then take an edge xy in G . By Menger's theorem, there exist at least $\kappa(G)$ internally vertex-disjoint xy -paths in G and thus, at least $\kappa(G) - 1$ of these are not the edge xy . Therefore, there is such a path with at most $\frac{n}{\kappa(G) - 1} + 2$ vertices, which together with the edge xy , then creates a cycle of length at most $\frac{n}{\kappa(G) - 1} + 2$. If $r \geq 1$, by the previous paragraph, G contains a P_{2r}^r – let x, y be its endpoints. By Menger's theorem, there exist at least $\kappa(G)$ internally vertex-disjoint xy -paths in G . Since at most $2r - 1$ of these intersect $P_{2r}^r \setminus \{x, y\}$, there exists one which is disjoint to $P_{2r}^r \setminus \{x, y\}$ and contains at most $\frac{n}{\kappa(G) - 2r + 1}$ internal vertices. This produces the desired C_l^r .

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We can also use this type of cycles to extend the celebrated Chvátal-Erdős theorem [11].

Theorem 4 (Chvátal-Erdős [11]). *If for a graph G we have that $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.*

Our result states that if the Chvátal-Erdős condition is satisfied, then we can find a Hamiltonian cycle with a certain number of triangles, depending on the discrepancy between the connectivity and the independence number.

Theorem 5. *Every n -vertex graph G such that $\kappa(G) \geq \alpha(G)$ contains a C_n^r with $r = \left\lfloor \frac{\kappa(G) - \alpha(G)}{2} \right\rfloor$.*

Proof. Suppose for contradiction that some $\ell < n$ is maximal such that there exists a copy of C_ℓ^r in G . Note that ℓ exists by Lemma 3. Order the cycle as $v_1 v_2 \dots v_\ell v_1$ so that the edges $v_1 v_3, v_3 v_5, \dots, v_{2r-1} v_{2r+1}$ are also present. Since $\ell < n$, there is a vertex v not in C_ℓ^r . Moreover, as $\kappa(G) \geq \alpha(G) + 2r$, there exist $\alpha(G)$ paths contained in $V(G) \setminus \{v_1, \dots, v_{2r}\}$, all of which go from v to C_ℓ^r and are vertex-disjoint apart from the initial vertex v . Let us denote these paths as $P_{i_1}, P_{i_2}, \dots$ so that $v_j = P_j \cap C_\ell^r$. Consider the set $S := \{v_{i_1+1}, v_{i_2+1}, \dots\}$ with indices taken modulo ℓ , so that $|S| \geq \alpha(G)$. Observe (as illustrated in Figure 1) that then there must be an edge contained in $S \cup \{v\}$ and that any such edge can be used to augment C_ℓ^r to a $C_{\ell'}^r$ with $\ell' > \ell$, contradicting the maximality of ℓ . $\square$

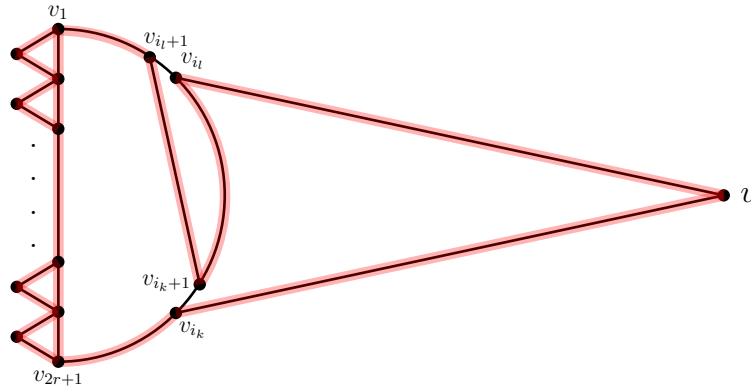


Figure 1: An illustration of how an edge between two elements v_{i_k+1}, v_{i_l+1} of S can be used to construct a new $C_{\ell'}^r$.

Now we show a result which uses only the minimum degree of the graph to shorten a path between two vertices. Among other shortening/augmenting tools in our paper, this is an important building block for our proof.

Lemma 6. *Let G be an n -vertex graph, $\delta := \delta(G)$ and P a path in G with endpoints x, y such that $|P| > 20n/\delta$. Then there is an xy -path P' such that $|P| - 20n/\delta \leq |P'| < |P|$.*

Proof. Suppose for sake of contradiction that no such path P' exists. Let $P := v_1 v_2 \dots v_{l-1} v_l$ with $v_1 = x, v_l = y$ and let $<_P$ denotes the given ordering of the path P as $v_1 <_P v_2 <_P \dots <_P v_l$. Since $|P| > 10n/\delta$, we can partition P into sub-paths $Q_1, Q_2, \dots, Q_k$ such that $|Q_k| \leq 10n/\delta$ and $|Q_i| = 10n/\delta$ for all $i < k$. Moreover, we have $k = \left\lceil \frac{|P|}{10n/\delta} \right\rceil$. Now, consider the vertices in Q_1 and take a subset $Q'_1 \subseteq Q_1$ of size $\lfloor |Q_1|/3 \rfloor \geq 3n/\delta$ such that no two vertices in Q'_1 are at distance at most 2 in P . Consider then the set of edges incident to Q'_1 , that is, $E[Q'_1, V(G)]$; by the minimum degree condition, there are at least $|Q'_1| \cdot \delta \geq 3n$ such edges.

Now, clearly there cannot exist an edge spanned by Q_1 which does not belong to P since this edge could be used to shorten P by at most $|Q_1| \leq 10n/\delta$. Hence, $e(Q'_1, Q_1) \leq 2|Q'_1|$. Similarly, the following must hold.

Claim. $e(Q'_1, V(G) \setminus P) \leq n - |P|$.

Proof. Suppose otherwise. Then there is a vertex $v \in V(G) \setminus P$ with at least 2 neighbours in Q'_1 - denote these by u, w . Note that since by construction u, w are at distance at least 2 and at most $|Q_1| \leq 10n/\delta$ in P , this is a contradiction, since it produces the desired P' by substituting the sub-path of P between u and w by the path uvw . $\square$

To give an upper bound on the total number of edges incident to Q'_1 which are contained in $V(P)$, we also use the following claim.

Claim. For all $i > 1$, we have $e(Q'_1, Q_i) < |Q'_1| + |Q_i|$.

Proof. Suppose otherwise. This implies that there is a cycle in $G[Q'_1, Q_i]$ and hence, there must exist two crossing edges in this bipartite graph, that is, edges $a_1 b_1$ and $a_2 b_2$, with $a_1 <_P a_2$ and both in Q'_1 , and $b_1 <_P b_2$ both in Q_i . These can clearly be used to shorten P (see Figure 2) by at most $|Q_1| + |Q_i| \leq 20n/\delta$, which is a contradiction as it produces the desired P' . $\square$

The above claim implies that

$$\sum_{i>1} e(Q'_1, Q_i) < \sum_{i>1} (|Q'_1| + |Q_i|) \leq (k-1)|Q'_1| + (|P| - |Q_1|) < 2|P| - 2|Q'_1|.$$



Figure 2: Shortening of the path P using the crossing edges $a_1 b_1$ and $a_2 b_2$. The resulting path is P' and is drawn in red.

To conclude, we now must have the following

$$e(Q'_1, V(G)) = e(Q'_1, Q_1) + e(Q'_1, V(G) \setminus P) + \sum_{i>1} e(Q'_1, Q_i) < 2|Q'_1| + (n - |P|) + (2|P| - 2|Q'_1|) < 2n.$$

which contradicts the previous observation that $e(Q'_1, V(G)) \geq 3n$. $\square$

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