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A generalization of Bondy's pancyclicity theorem

(Extended abstract)

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Abstract

The *bipartite independence number* of a graph G, denoted as $\tilde{\alpha}(G)$, is the minimal number k such that there exist positive integers a and b with $a + b = k + 1$ with the property that for any two sets $A, B \subseteq V(G)$ with $|A| = a$ and $|B| = b$, there is an edge between A and B. McDiarmid and Yolov showed that if $\delta(G) \geq \tilde{\alpha}(G)$ then G is Hamiltonian, extending the famous theorem of Dirac which states that if $\delta(G) \geq |G|/2$ then G is Hamiltonian. In 1973, Bondy showed that, unless G is a complete bipartite graph, Dirac's Hamiltonicity condition also implies pancyclicity, i.e., existence of cycles of all the lengths from 3 up to n. In this paper we show that $\delta(G) \geq \tilde{\alpha}(G)$ implies that G is pancyclic or that $G = K_{\frac{n}{2},\frac{n}{2}}$, thus extending the result of McDiarmid and Yolov, and generalizing the classic theorem of Bondy.

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1 Introduction

The notion of Hamiltonicity is one of most central and extensively studied topics in Combinatorics. Since the problem of determining whether a graph is Hamiltonian is NP-complete, a central theme in Combinatorics is to derive sufficient conditions for this property. A classic example is Dirac's theorem [14] which dates back to 1952 and states that every n-vertex graph with minimum degree at least $n/2$ is Hamiltonian. Since then, a plethora of interesting and important results about various aspects of Hamiltonicity have been obtained, see e.g. [1, 11, 12, 13, 18, 24, 26, 27, 32], and the surveys [20, 29].

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Besides finding sufficient conditions for containing a Hamilton cycle, significant attention has been given to conditions which force a graph to have cycles of other lengths. Indeed, *the cycle spectrum of a graph*, which is the set of lengths of cycles contained in that graph, has been the focus of study of numerous papers and in particular gained a lot of attention in recent years [2, 3, 8, 16, 19, 21, 23, 28, 31, 34]. Among other graph parameters, the relation of the cycle spectrum to the minimum degree, number of edges, independence number, chromatic number and expansion of the graph have been studied.

We say that an n-vertex graph is *pancyclic* if the cycle spectrum contains all integers from 3 up to n. Bondy suggested that in the cycle spectrum of a graph, it is usually hardest to guarantee the existence of the longest cycle, i.e. a Hamilton cycle. This intuition was captured by his famous meta-conjecture [5] from 1973, which asserts that any non-trivial condition which implies Hamiltonicity, also implies pancyclicity (up to a small class of exceptional graphs). As a first example, he proved in [6] an extension of Dirac's theorem, showing that minimum degree at least $n/2$ implies that the graph is either pancyclic or that it is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Further, Bauer and Schmeichel [4], relying on previous results of Schmeichel and Hakimi [33], showed that the sufficient conditions for Hamiltonicity given by Bondy [7], Chvátal [10] and Fan [17] all imply pancyclicity, up to a certain small family of exceptional graphs.

Another classic Hamiltonicity result is the Chvátal-Erdős theorem, which states that $\kappa(G) \geq \alpha(G)$ implies that G is Hamiltonian, where $\kappa(G)$ is the connectivity of G, and $\alpha(G)$ its independence number. Motivated by Bondy's meta-conjecture, Jackson and Ordaz [22] thirty years ago suggested that $\kappa(G) > \alpha(G)$ already implies pancyclicity. The first progress towards this problem was obtained by Keevash and Sudakov, who showed pancyclicity when $\kappa(G) \geq 600\alpha(G)$. Recently, in [15] we were able to resolve the Jackson-Ordaz conjecture asymptotically, proving that $\kappa(G) \geq (1 + o(1))\alpha(G)$ is already enough for pancyclicity. It is worth mentioning that, in all the listed work, the proof that the Hamiltonicity condition also implies pancyclicity is usually significantly harder than just proving Hamiltonicity, and requires new ideas and techniques.

An interesting sufficient condition for Hamiltonicity was given by McDiarmid and Yolov [30]. To state their result, we need the following natural graph parameter. For a graph G, its *bipartite independence number* $\tilde{\alpha}(G)$ is the minimal number k, such that there exist positive integers a and b with $a + b = k + 1$, such that between any two sets $A, B \subseteq V(G)$ with $|A| = a$ and $|B| = b$, there is an edge between A and B. Notice that we always have that $\alpha(G) \leq \tilde{\alpha}(G)$. Indeed, if $\tilde{\alpha}(G) = k$, then G does not contain independent sets I of size at least $k + 1$, since evidently for every $a + b = k + 1$, there would exist disjoint sets $A, B \subset I$, so that $|A| = a$ and $|B| = b$ and with no edge between A and B. Let us now state the result of McDiarmid and Yolov.

Theorem 1 ([30])**.** *If $\delta(G) \geq \tilde{\alpha}(G)$, then G is Hamiltonian.*

This result implies Dirac's theorem, because if $\delta(G) \geq n/2$, then $\lceil n/2 \rceil \geq \tilde{\alpha}(G)$, as for every $|A| = 1$ and $|B| = \lceil n/2 \rceil$ there is an edge between A and B. Hence also $\delta(G) \geq \lceil n/2 \rceil \geq \tilde{\alpha}(G)$, so G is Hamiltonian.

Naturally, the immediate question which arises is whether the McDiarmid-Yolov condition implies that the graph satisfies the stronger property of pancyclicity. As a very preliminary step in this direction, Chen [9] was able to show that for any given positive constant c, for sufficiently large n it holds that if G is an n-vertex graph with $\tilde{\alpha}(G) = cn$ and $\delta(G) \geq \frac{10}{3}cn$, then G is pancyclic. In this paper we completely resolve this problem, showing that $\delta(G) \geq \tilde{\alpha}(G)$ implies that G pancyclic or $G = K_{\frac{n}{2},\frac{n}{2}}$. This generalizes the classical theorem of Bondy [6], and gives additional evidence for his meta-conjecture, mentioned above.

Theorem 2. *If $\delta(G) \geq \tilde{\alpha}(G)$, then G is pancyclic, unless G is complete bipartite $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Our proof is completely self-contained and relies on a novel variant of Pósa's celebrated rotation-extension technique, which is used to extend paths and cycles in expanding graphs (see, e.g., [32]). Define the graph $\tilde{C}_\ell$, to be the cycle of length ℓ together with an additional vertex which is adjacent to two consecutive vertices on the cycle (thus forming a triangle with them). For each $\ell \in [3, n-1]$, our goal is to either find a $\tilde{C}_\ell$ or a $\tilde{C}_{\ell+1}$, which is clearly enough to show pancyclicity. The proof is recursive in nature, as we will derive the existence of a $\tilde{C}_\ell$ or a $\tilde{C}_{\ell+1}$ from the existence of a $\tilde{C}_{\ell-1}$. In our setting, we would like to apply the rotation-extension technique to the $\tilde{C}_{\ell-1}$ with the additional requirement that the extended cycle preserves the attached triangle. However, this is not possible in general and from the existence of a $\tilde{C}_{\ell-1}$ we will in turn derive the existence of a gadget denoted as a *switch*, which is a path with triangles attached to it, to which we can apply our rotation-extension technique. One of the key ideas is to consider the switch which is optimal with respect to how close the triangles are to the beginning of the path. The application of the rotation-extension technique to such an optimal switch will then result in either a $\tilde{C}_\ell$, a $\tilde{C}_{\ell+1}$, or a better switch, contradicting the optimality of the original switch.

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